USA Mathematical Talent Search
CREDITS and QUICK ANSWERS
Round 4 — Year 15 — Academic Year 2003–2004

1/4/15. Find, with proof, the smallest positive integer n for which the sum of the digits of $29n$ is as small as possible.

This problem was fashioned after Problem 3 of the 2nd Lithuanian Olympiad administered at Vilnius University on September 30, 2000.

We search for $29n$ rather than directly for n. Numbers with only a single nonzero digit are that digit times a power of ten, which would not be divisible by 29. Of the numbers with two nonzero digits, the smallest digit sum occurs with both nonzero digits being 1. The smallest such number must begin and end in 1, because if it ended in zero, we could divide by 10. So we are looking for the smallest integer of the form $10^e + i$ that is a multiple of 29.

The powers of 10 mod 29 are 1, 10, 13, 14, 24, 8, 22, 17, 25, 18, 6, 2, 20, 26, 28, 19, 16, 15, 5, 21, 7, 12, 4, 11, 23, 27, 9, 3, 1, ... so $10^{14} \equiv 28 \pmod{29}$. Therefore $10^{14} + 1$ is divisible by 29, and $n = (10^{14} + 1)/29 = 3448275862069$.

2/4/15. For four integer values of n greater than six, there exist right triangles whose side lengths are integers equivalent to 4, 5, and 6 modulo n, in some order. Find those values. Prove that at most four such values exist. Also, for at least one of those values of n, provide an example of such a triangle.

This problem was devised by Dr. Peter Anspach of the NSA, after a lunchtime conversation among mathematicians about trying to create a $(4, 5, 6)$ right triangle.

Let a, b, and c be the sides of such a right triangle, with $a^2 + b^2 = c^2$. If $c \equiv 4 \pmod{n}$, then $5^2 + 6^2 \equiv 4^2 \pmod{n}$ so n is a factor of $5^2 + 6^2 - 4^2 = 45$. If $c \equiv 5 \pmod{n}$, then n is a factor of $4^2 + 6^2 - 5^2 = 27$. If $c \equiv 6 \pmod{n}$, then n is a factor of $4^2 + 5^2 - 6^2 = 5$, but since $n > 6$, we can reject this case. Thus, n is a factor of 27 or 45, so n is 9, 15, 27, or 45.

Given an n, a triple (a, b, c) that forms a right triangle can be found by trial and error, by applying number theory to an equation such as $(9x + 5)^2 + (9y + 6)^2 = (9z + 4)^2$, or by the $(r(s^2 - t^2), 2rst, r(s^2 + t^2))$ method of finding Pythagorean triples after solving for r, s, and t that give the right values mod n. The first three solutions for each case are:

for $n = 9$ and $c \equiv 4 \pmod{9}$: $(24, 32, 40)$, $(51, 68, 85)$, $(78, 104, 130)$;

for $n = 9$ and $c \equiv 5 \pmod{9}$: $(40, 96, 104)$, $(15, 112, 113)$, $(60, 175, 185)$;

for $n = 15$ and $c \equiv 4 \pmod{15}$: $(65, 156, 169)$, $(140, 336, 364)$, $(215, 516, 559)$;

for $n = 27$ and $c \equiv 5 \pmod{27}$: $(168, 490, 518)$, $(33, 544, 545)$, $(814, 1248, 1490)$;

for $n = 45$ and $c \equiv 4 \pmod{45}$: $(1176, 2975, 3199)$, $(3920, 16926, 17374)$, $(1760, 17556, 17644)$.

3/4/15. Find a nonzero polynomial $f(w, x, y, z)$ in the four indeterminates w, x, y, and z of minimum degree such that switching any two indeterminates in the polynomial gives the same polynomial except that its sign is reversed. For example, $f(z, x, y, w) = -f(w, x, y, z)$. Prove that the degree of the polynomial is as small as possible.

This problem was invented by Dr. David Grabiner, an NSA mathematician who helps with the evaluation and grading of problems for the USAMTS.

Consider any two of the indeterminants, say w and x. If the standard form of $f(w, x, y, z)$ has a term $kw^a x^b y^c z^d$, then it must have a matching term $-kw^b x^a y^c z^d$. Grouping them together gives either $k(w^{a-b} - x^{a-b})w^b x^b y^c z^d$ or $k(x^{b-a} - w^{b-a})w^a x^a y^c z^d$, which both are divisible by $w - x$. Since the entire polynomial can be grouped this way, it is divisible by $w - x$.

Thus, for every pair of indeterminants, the polynomial is divisible by their difference. This means $f(w, x, y, z)$ is divisible by $(w-x)(w-y)(w-z)(x-y)(x-z)(y-z)$. Letting $f(w, x, y, z) = (w-x)(w-y)(w-z)(x-y)(x-z)(y-z)$ keeps the degree to a minimum, and it works because switching any two indeterminants in $(w-x)(w-y)(w-z)(x-y)(x-z)(y-z)$ gives the same polynomial except that its sign is reversed.

4/4/15. For each nonnegative integer n define the function $f_n(x)$ by

$$f_n(x) = \sin^n(x) + \sin^n(x + \frac{2\pi}{3}) + \sin^n(x + \frac{4\pi}{3})$$

for all real numbers x, where the sine functions use radians. The functions $f_n(x)$ can be also expressed as polynomials in $\sin(3x)$ with rational coefficients. For example,

$$f_0(x) = 3, \qquad f_1(x) = 0, \qquad f_2(x) = \frac{3}{2}, \qquad f_3(x) = -\frac{3}{4}\sin(3x),$$
$$f_4(x) = \frac{9}{8}, \qquad f_5(x) = -\frac{15}{16}\sin(3x), \qquad f_6(x) = \frac{27}{32} + \frac{3}{16}\sin^2(3x),$$

for all real numbers x. Find an expression for $f_7(x)$ as a polynomial in $\sin(3x)$ with rational coefficients, and prove that it holds for all real numbers x.

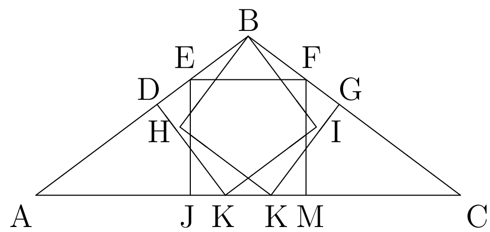
This problem was devised by the distinguished Hungarian mathematician and poet Mihály Bencze of Brasso, Transylvania.

The triple angle formula for sine is $\sin(3x) = 3\sin(x) - 4\sin^3(x)$. But since $\sin(3x)$ also equals $\sin(3(x + \frac{2\pi}{3}))$ and $\sin(3(x + \frac{4\pi}{3}))$, it also expands to $3\sin(x + \frac{2\pi}{3}) - 4\sin^3(x + \frac{2\pi}{3})$ and $3\sin(x + \frac{4\pi}{3}) - 4\sin^3(x + \frac{4\pi}{3})$. Thus, for all real numbers x,

$$\begin{aligned}
f_n(x)\sin(3x) &= \sin^n(x)\sin(3x) + \sin^n(x + \frac{2\pi}{3})\sin(3x) + \sin^n(x + \frac{4\pi}{3})\sin(3x) \\
&= \sin^n(x)\big(3\sin(x) - 4\sin^3(x)\big) \\
&\quad + \sin^n(x + \frac{2\pi}{3})\big(3\sin(x + \frac{2\pi}{3}) - 4\sin^3(x + \frac{2\pi}{3})\big) \\
&\quad + \sin^n(x + \frac{4\pi}{3})\big(3\sin(x + \frac{4\pi}{3}) - 4\sin^3(x + \frac{4\pi}{3})\big) \\
&= 3\big(\sin^{n+l}(x) + \sin^{n+l}(x + \frac{2\pi}{3}) + \sin^{n+l}(x + \frac{4\pi}{3})\big) \\
&\quad - 4\big(\sin^{n+3}(x) + \sin^{n+3}(x + \frac{2\pi}{3}) + \sin^{n+3}(x + \frac{4\pi}{3})\big) \\
&= 3f_{n+l}(x) - 4f_{n+3}(x).
\end{aligned}$$

Setting n to 4 and rearranging gives $f_7(x) = \frac{3}{4}f_5(x) - \frac{1}{4}f_4(x)\sin(3x) = (\frac{3}{4})(-\frac{15}{16})\sin(3x) - (\frac{1}{4})(\frac{9}{8})\sin(3x) = (-\frac{63}{64})\sin(3x)$.

5/4/15. Triangle ABC is an obtuse isosceles triangle with the property that three squares of equal size can be inscribed in it as shown on the right. The ratio AC/AB is an irrational number that is the root of a cubic polynomial. Determine that polynomial.



This triangle was apparently discovered by the Italian mathematician Eugenio Calabi. It is the only non-equilateral triangle into which we can fit three equal squares in this manner. The triangle is displayed on page 266 of *The Book of Numbers,* by John H. Conway and Richard K. Guy, published by Springer-Verlag in 1996.

Let s be the length of a side of the squares. By symmetry, $AJ = MC$, so $2(AJ) = AC - JM = AC - s$. $AD = AB - BD = AB - s$. Triangle AEJ is congruent to triangle AKD, so $AD = AJ$. So $AC - s = 2(AB - s)$, giving $s = 2(AB) - AC$ and $AJ = AC - AB$.
Let P be the midpoint of side $\overline{AC}$. Triangle AEJ is similar to triangle ABP. So $AE/AJ = AB/AP = 2(AB/AC)$. $AE = 2(AC - AB)(AB/AC)$.
By the Pythagorean Theorem $(AE)^2 = (AJ)^2 + s^2$, which converts to

$$\begin{aligned}\left(2(AC - AB)\left(\frac{AB}{AC}\right)\right)^2 &= (AC - AB)^2 + (2(AB) - AC)^2 &(1)\\ (2(AC - AB)(AB))^2 &= (AC)^2(AC - AB)^2 + (AC)^2(2(AB) - AC)^2. &(2)\end{aligned}$$

Let $x = AC/AB$. Equation (2) converts to

$$\begin{aligned}4(x-1)^2 &= x^2(x-1)^2 + x^2(2-x)^2\\ (4 - x^2)(x-1)^2 - x^2(2-x)^2 &= 0\\ (2-x)(x+2)(x-1)^2 - x^2(2-x)^2 &= 0\\ (2-x)\big((x+2)(x-1)^2 - x^2(2-x)\big) &= 0\\ (2-x)(2x^3 - 2x^2 - 3x + 2) &= 0\end{aligned}$$

By the triangle inequality, $AC < AB + BC = 2(AB)$, so $x < 2$ and $2 - x \neq 0$. Therefore, AC/AB is the root of the cubic polynomial $2x^3 - 2x^2 - 3x + 2$.