USA Mathematical Talent Search
PROBLEMS
Round 4 — Year 15 — Academic Year 2003–2004

1/4/15. Find, with proof, the smallest positive integer n for which the sum of the digits of $29n$ is as small as possible.

2/4/15. For four integer values of n greater than six, there exist right triangles whose side lengths are integers equivalent to 4, 5, and 6 modulo n, in some order. Find those values. Prove that at most four such values exist. Also, for at least one of those values of n, provide an example of such a triangle.

3/4/15. Find a nonzero polynomial $f(w, x, y, z)$ in the four indeterminates w, x, y, and z of minimum degree such that switching any two indeterminates in the polynomial gives the same polynomial except that its sign is reversed. For example, $f(z, x, y, w) = -f(w, x, y, z)$. Prove that the degree of the polynomial is as small as possible.

4/4/15. For each nonnegative integer n define the function $f_n(x)$ by

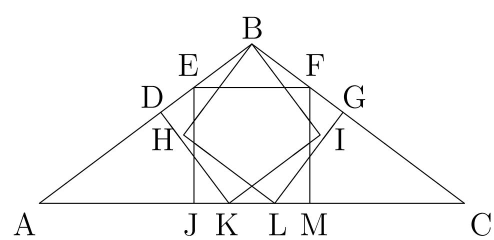
$$f_n(x) = \sin^n(x) + \sin^n(x + \frac{2\pi}{3}) + \sin^n(x + \frac{4\pi}{3})$$

for all real numbers x, where the sine functions use radians. The functions $f_n(x)$ can be also expressed as polynomials in $\sin(3x)$ with rational coefficients. For example,

$$f_0(x) = 3, \qquad f_1(x) = 0, \qquad f_2(x) = \frac{3}{2}, \qquad f_3(x) = -\frac{3}{4}\sin(3x),$$
$$f_4(x) = \frac{9}{8}, \qquad f_5(x) = -\frac{15}{16}\sin(3x), \qquad f_6(x) = \frac{27}{32} + \frac{3}{16}\sin^2(3x),$$

for all real numbers x. Find an expression for $f_7(x)$ as a polynomial in $\sin(3x)$ with rational coefficients, and prove that it holds for all real numbers x.

5/4/15. Triangle ABC is an obtuse isosceles triangle with the property that three squares of equal size can be inscribed in it as shown on the right. The ratio AC/AB is an irrational number that is the root of a cubic polynomial. Determine that polynomial.



Complete, well-written solutions to at least two of the problems above, accompanied by a Cover Sheet, should be mailed to:

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and postmarked by Sunday, 14 March 2004. Each participant is expected to develop solutions without help from others. For the cover sheet and other details, see the USAMTS web site: http://www.nsa.gov/programs/mepp/usamts.html.