USA Mathematical Talent Search
CREDITS and QUICK ANSWERS
Round 3 — Year 15 — Academic Year 2003–2004

1/3/15. Find, with proof, all pairs of two-digit positive integers ab and cd such that all the digits a, b, c, and d are different from one another and $(ab)(cd) = (dc)(ba)$.

This is an old problem that Professor Emeritus George Berzenyi, the founder of the USAMTS, remembered for this contest without recalling its original source.

The numerals in $(ab)(cd) = (dc)(ba)$ expand out as

$$\begin{aligned} (10a+b)(10c+d) &= (10d+c)(10b+a) \\ 100ac+10ad+10bc+bd &= 100bd+10ad+10bc+ac \\ 99ac &= 99bd \\ ac &= bd. \end{aligned}$$

So we need only look for all pairs of distinct digits whose products are the same, $(1)(6) = (2)(3)$, $(1)(8) = (2)(4)$, $(2)(6) = (3)(4)$, $(2)(9) = (3)(6)$, and $(3)(8) = (4)(6)$, and arrange them in all possible orders:

$$\begin{array}{ll} (12)(63) = (36)(21) & (13)(62) = (26)(31) \\ (12)(84) = (48)(21) & (14)(82) = (28)(41) \\ (23)(64) = (46)(32) & (24)(63) = (36)(42) \\ (23)(96) = (69)(32) & (26)(93) = (39)(62) \\ (34)(86) = (68)(43) & (36)(84) = (48)(63). \end{array}$$

2/3/15. Find the smallest positive integer n such that the product $(2004n+1)(2008n+1)$ is a perfect square. Prove that n is as small as possible.

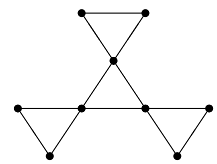
This problem was devised by Professor Emeritus George Berzsenyi and modified to be more challenging by Dr. David Grabiner, a mathematician at the National Security Agency.

Since $501(2004n+1) - 500(2008n+1) = 1$, the two factors $2004n+1$ and $2008n+1$ are relatively prime. So both factors must be perfect squares. Let $x^2 = 2004n+1$ and $y^2 = 2008n+1$.

Since both x and y are odd integers, and y is larger than x, we have $y \geq x+2$. So $y^2 - x^2 \geq (x+2)^2 - x^2 = 4x+4$. We also have that $y^2 - x^2 = (2008n+1) - (2004n+1) = 4n$. Thus, $x+1 \leq n$. Subtracting 2 from both sides of the inequality gives $x-1 \leq n-2$. The values $x+1$, $x-1$, n, and $n-2$ are positive, since x is at least $\sqrt{2004(1)+1} = 44.78$, so we can multiply the inequalities to get $(x+1)(x-1) \leq n(n-2)$. Because $(x+1)(x-1) = x^2 - 1 = 2004n$, this gives $2004n \leq n(n-2)$, so $2004 \leq n-2$ and $2006 \leq n$.

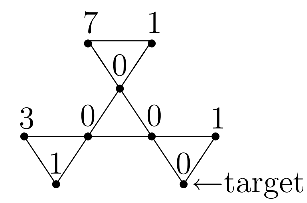
Trying $n = 2006$ gives $(2004n+1)(2008n+1) = (2005^2)(2007^2) = 4024035^2$.

3/3/15. Pebbles are put on the vertices of a combinatorial graph. For a vertex with two or more pebbles, a *pebbling step* at that vertex removes one pebble at the vertex from the graph entirely and moves another pebble at that vertex to a chosen adjacent vertex. The *pebbling number* of a graph is the smallest number t such that no matter how t pebbles are distributed on the graph, the distribution would have the property that for every empty vertex a series of pebbling steps could move a pebble to that one vertex. For example, the pebbling number of the graph formed from the vertices and edges of a hexagon is eight. Find, with proof, the pebbling number of the graph illustrated on the right.

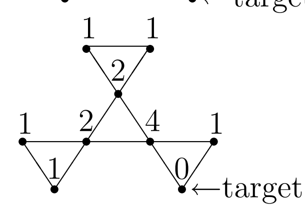


Pebbling was invented in 1987 to simplify a number theoretic proof. Check the work of Glenn Hurlbert at the Southeastern Combinatorics/Graph Theory Conference at Florida Atlantic University for details. Dr. Michelle Wagner of the NSA brought pebbling to the attention of the USAMTS. Dr. Erin Schram of the NSA chose the graph, which, incidentally, is the graph of the states of a hexahexaflexagon.

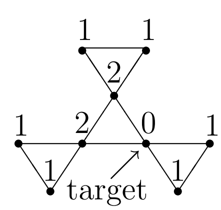
The pebbling number is more than 13, since the 13-pebble distribution illustrated on the right cannot move a pebble to the target vertex in a series of pebbling steps. Thus, we want to see whether 14 pebbles will always be enough to put a pebble on a target vertex.



If the target vertex is one of the corner vertices, assign a point value to each vertex of the graph, as illustrated on the right. Each pebble on the graph is worth the point value of its vertex. The points are designed so that whenever a pebbling step moves a pebble toward the target vertex, the total point value of all pebbles on the graph does not change unless we move a pebble onto the target vertex. Also, the total point value of one pebble on each of the eight vertices beside the target vertex is 13. Therefore, whenever the total point value of all the pebbles is more than 13, some vertex besides the target vertex has two or more pebbles and we can make a pebbling step to move a pebble toward the target vertex. Suppose we start with 14 pebbles. We would either start with a pebble on the target vertex or start with 14 or more total points. With a total of 14 or more points, we can repeatedly make pebbling steps toward the target vertex until we finally move a pebble onto the target vertex.



If the target vertex is one of the central vertices, use the point system illustrated on the right instead. We can always make a pebbling step toward the target vertex if we have 11 or more points. Since 14 pebbles give us 14 or more points unless a pebble is on the target vertex, that means that 14 pebbles give us enough to move a pebble onto that target vertex.



Therefore, the pebbling number of the graph is 14.

4/3/15. A infinite sequence of quadruples begins with the five quadruples $(1,3,8,120)$, $(2,4,12,420)$, $(3,5,16,1008)$, $(4,6,20,1980)$, $(5,7,24,3432)$. Each quadruple (a,b,c,d) in this sequence has the property that the six numbers $ab+1$, $ac+1$, $bc+1$, $ad+1$, $bd+1$, and $cd+1$ are all perfect squares. Derive a formula for the nth quadruple in the sequence and demonstrate that the property holds for every quadruple generated by the formula.

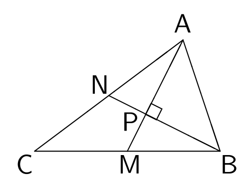
This problem was inspired by some of the work of Neven Juric of Croatia.

The quadruples are given by the formula $(n, n+2, 4n+4, 16n^3+48n^2+44n+12)$.

The six perfect squares are

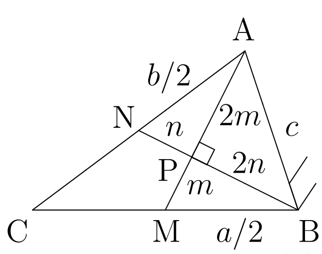
$$\begin{aligned}
(n)(n+2)+1 &= n^2+2n+1=(n+1)^2,\\
(n)(4n+4)+1 &= 4n^2+4n+1=(2n+1)^2,\\
(n+2)(4n+4)+1 &= 4n^2+12n+9=(2n+3)^2,\\
(n)(16n^3+48n^2+44n+12)+1 &= 16n^4+48n^3+44n^2+12n+1\\
&= (4n^2+6n+1)^2,\\
(n+2)(16n^3+48n^2+44n+12)+1 &= 16n^4+80n^3+140n^2+100n+25\\
&= (4n^2+10n+5)^2,\\
(4n+4)(16n^3+48n^2+44n+12)+1 &= 64n^4+256n^3+368n^2+224n+49\\
&= (8n^2+16n+7)^2.
\end{aligned}$$

5/3/15. In triangle ABC the lengths of the sides of the triangle opposite to the vertices A, B, and C are known as a, b, and c, respectively. Prove there exists a constant k such that if the medians emanating from A and B are perpendicular to one another, then $a^2+b^2=kc^2$. Also find the value of k.



Prof. George Berzsenyi invented this problem, but he doubts that he was the first to come up with it.

Let m denote the length MP and n denote the length NP. Since the medians of a triangle intersect one third of their length from their base end, this gives that $AP=2m$ and $BP=2n$. The perpendicular medians create three right triangles with the sides of triangle ABC and we can use the Pythagorean theorem to find relations between the medians and the sides:



$\triangle BMP$ gives $m^2+(2n)^2=(a/2)^2$, so $a^2=4m^2+16n^2$ (1)

$\triangle ANP$ gives $(2m)^2+n^2=(b/2)^2$, so $b^2=16m^2+4n^2$ (2)

$\triangle ABP$ gives $(2m)^2+(2n)^2=c^2$, so $c^2=4m^2+4n^2$ (3)

Summing equations (1) and (2) gives $a^2+b^2=20m^2+20n^2$. Since $c^2=4m^2+4n^2$, this means $a^2+b^2=5c^2$. The value of k is 5.