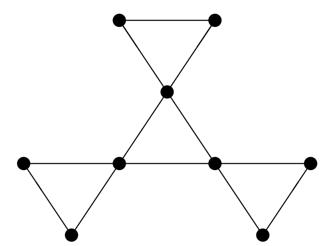
USA Mathematical Talent Search
PROBLEMS
Round 3 — Year 15 — Academic Year 2003–2004

1/3/15. Find, with proof, all pairs of two-digit positive integers ab and cd such that all the digits a, b, c, and d are different from one another and $(ab)(cd) = (dc)(ba)$.

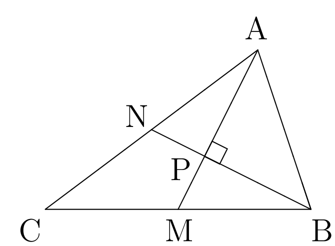
2/3/15. Find the smallest positive integer n such that the product $(2004n + 1)(2008n + 1)$ is a perfect square. Prove that n is as small as possible.

3/3/15. Pebbles are put on the vertices of a combinatorial graph. For a vertex with two or more pebbles, a *pebbling step* at that vertex removes one pebble at the vertex from the graph entirely and moves another pebble at that vertex to a chosen adjacent vertex. The *pebbling number* of a graph is the smallest number t such that no matter how t pebbles are distributed on the graph, the distribution would have the property that for every empty vertex a series of pebbling steps could move a pebble to that one vertex. For example, the pebbling number of the graph formed from the vertices and edges of a hexagon is eight. Find, with proof, the pebbling number of the graph illustrated on the right.



4/3/15. An infinite sequence of quadruples begins with the five quadruples $(1, 3, 8, 120)$, $(2, 4, 12, 420)$, $(3, 5, 16, 1008)$, $(4, 6, 20, 1980)$, $(5, 7, 24, 3432)$. Each quadruple (a, b, c, d) in this sequence has the property that the six numbers $ab + 1$, $ac + 1$, $bc + 1$, $ad + 1$, $bd + 1$, and $cd + 1$ are all perfect squares. Derive a formula for the nth quadruple in the sequence and demonstrate that the property holds for every quadruple generated by the formula.

5/3/15. In triangle ABC the lengths of the sides of the triangle opposite to the vertices A, B, and C are known as a, b, and c, respectively. Prove there exists a constant k such that if the medians emanating from A and B are perpendicular to one another, then $a^2 + b^2 = kc^2$. Also find the value of k.



Complete, well-written solutions to at least two of the problems above, accompanied by a Cover Sheet, should be mailed to:

USA Mathematical Talent Search
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279 East Central Street, Suite 246
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and postmarked by Sunday, 4 January 2004. Each participant is expected to develop solutions without help from others. For the cover sheet and other details, see the USAMTS web site: http://www.nsa.gov/programs/mepp/usamts.html.