USA Mathematical Talent Search
PROBLEMS
Round 1 — Year 15 — Academic Year 2003–2004

1/1/15. Let $f(x) = \lfloor x \rfloor + \lfloor 10x \rfloor + \lfloor 100x \rfloor + \lfloor 1000x \rfloor$ for all real numbers x, where for any real number r, $\lfloor r \rfloor$ means the greatest integer less than or equal to r. For example, $f(\pi) = 3 + 31 + 314 + 3141 = 3489$. Find, with proof, a positive integer n less than 2003 such that $f(x) = n$ has no solution for x, but $f(y) = n + 11$ and $f(z) = n + 111$ have solutions for y and z.

2/1/15. Find all primes p for which $6p + 1$ is the fifth power of an integer. Prove that you found all of them.

3/1/15. We attempted to arrange the integers 1, 2, 3, ..., 12 around a circle so that all sums of pairs of adjacent integers are either

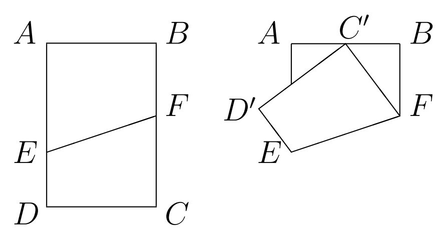
(a) all perfect squares, or

(b) all triangular numbers, which are numbers of the form $n(n + 1)/2$.

One case out of (a) and (b) succeeded. Which one? For the successful case, show a valid arrangement. For the unsuccessful case, explain why it is impossible.

4/1/15. Two players play a game. Each player, in turn, has to name a positive integer that is less than the previous number but at least half the previous number. The player who names the number 1 loses. If the first player starts by naming 2003 and after that both players play with the the best strategy, who wins? Describe the strategy and prove it works.

5/1/15. A rectangular piece of paper, $ABCD$, is folded along line segment $\overline{EF}$, where point E is on side $\overline{AD}$ and point F is on side $\overline{BC}$, so that C ends up at the midpoint of side $\overline{AB}$. Determine the length of $\overline{EF}$ if the length of $\overline{AB}$ is 240 and the length of $\overline{BC}$ is 288.



Complete, well-written solutions to at least two of the problems above, accompanied by an entry form and cover sheet, should be mailed to the administrative address listed on the USAMTS web site and postmarked no later than **5 October 2003**. Each participant is expected to develop solutions without help from others. For the cover sheet and other details, see the USAMTS web site: http://www.nsa.gov/programs/mepp/usamts.html.