USA Mathematical Talent Search

PROBLEMS / SOLUTIONS / COMMENTS

Round 4 - Year 14 - Academic Year 2002-2003

solutions edited by Dr. Erin Schram

1/4/14. The sequence of letters **TAGC** is written in succession 55 times on a strip, as shown below. The strip is to be cut into segments between letters, leaving strings of letters on each segment, which we will call words. For example, a cut after the first **G**, after the second **T**, and after the second **C** would yield the words **TAG**, **CT**, and **AGC**. At most how many distinct words could be found if the entire strip were cut? Justify your answer.

T A G C T A G C T A G · · · C T A G C

Comment: This problem was inspired by Problem 1759 of Sándor Róka's excellent collection of 2000 problems in elementary mathematics (*2000 Feladat az Elemi Matematika Köréböl,* Typotex, Budapest, 2000).

Solution 1 to 1/4/14 by Becky Wright (9/UT):

I will assume that a word must have at least one letter. If that is not the case, please add one word to my total to account for the one word with no letters in it.

I claim that the maximum number of distinct words is 40.

Forty distinct words can be constructed as follows. Make three cuts to cut the original strip of 220 letters into four consecutive segments containing 55 letters each. Next cut each of these segments in the same way: make the first cut after the first letter, make the second cut after two more letters, make the third cut after three more letters, and so on until the ninth cut after nine more letters. The segment after the last cut has ten letters, since $1 + 2 + 3 + \ldots + 9 + 10 = 55$. So each segment of 55 letters has been cut into ten words, and this gives a total of 40 words from the original strip of 220 letters. Next, note that these 40 words contain no repetitions. These 40 words contain four words of length one, four words of length two, and so on up to four words of length ten. For any given length, the four words of that length are offset from the previous word of that length by 55 letters. Since $55 \equiv 3 \pmod{4}$, then for a given length, the second word of that length will start three letters later in the sequence than the first word, the third word will be six (which is the same as two) letters later in the sequence, and the fourth word will be nine (which is the same as one) letters later in the sequence. Therefore, the four words must start with different letters and are distinct.

To see that no more than 40 words is possible, consider the following. First of all, there are at most four possible distinct words of a given length. The four letters repeat in the same order everywhere on the strip, so the starting letter and the word length completely determine the word. Since there are only four possible starting letters, this forces no more than four possible words of a given length. The way to construct the largest collection of words is to use words of shortest length as much as possible, which is four times for each length. So the 40 shortest distinct words use four words each of lengths one through ten. This uses exactly 220 letters. Hence, any collection of more than 40 distinct words will have to use more than 220 letters, so 40 is the maximum.

Solution 2 for 1/4/14 by Alan Gostin (11/TX):

There are four distinct words for each length of word, one starting with each of the four letters. Obviously, since the most words are wanted, it is best to use the shortest words possible and not to waste any words on repeating words. The 40 smallest words, four of each length from one to ten, add to 220 letters. Thus, the largest number of words that might be possible is 40.

However, it is as of yet uncertain whether these words can all be fit into the 220 letters. The solution is simple, though. By cutting one string of each of the ten lengths off the beginning, ten distinct strings are created using 55 letters. If this is repeated, taking special care to cut in the same order as the first cutting, ten more strings are obtained. These strings must be different from the first ten, because instead of starting with the first letter T, this set of words starts with the 56th letter C. The first letter of each new set must be different because the greatest common factor of 55 and 4 is 1. By following this method of cutting, 40 distinct strings may be created, as illustrated by the example below.

Example cutting (First set of ten words on the first line)
T AG CTA GCTA GCTAG CTAGCT AGCTAGC TAGCTAGC TAGCTAGCT AGCTAGCTAG
C TA GCT AGCT AGCTA GCTAGC TAGCTAG CTAGCTAG CTAGCTAGC TAGCTAGCTA
G CT AGC TAGC TAGCT AGCTAG CTAGCTA GCTAGCTA GCTAGCTAG CTAGCTAGCT
A GC TAG CTAG CTAGC TAGCTA GCTAGCT AGCTAGCT AGCTAGCTA GCTAGCTAGC

Solution 3 for 1/4/14 by Gleb Kuznetsov (9/UT):

By analyzing the given series of letters, it is evident that since the order of letters doesn't change, there are only four distinct words for each size of word, each starting with a different letter.

By listing out the letters, and marking where the strip would be cut, I found that alternating between the four words of an odd size and the four words of an even size creates distinct words of each length. Once all combinations of those sizes are used, a new pair of sizes is used; *i.e.*, after all 1 and 2 lengths are cut, move on to 3 and 4, and so on.

Since there are $4 \times 55 = 220$ letters on the strip, the pattern can be continued until that many letters are used. And since there are four possibilities for each word size, the number of letters used to make all words up to that size can be calculated by $4(1 + 2 + 3 + \ldots)$, increasing the number of terms in the sum until all 220 letters are used. Since $4(1 + 2 + 3 + \ldots + 10) = 220$, ten word sizes are used, and 40 distinct words are made.

1&2**: T|AG|C|TA|G|CT|A|GC|**
3&4: **TAG|CTAG|CTA|GCTA|GCT|AGCT|AGC|TAGC|**
5&6: **TAGCT|AGCTAG|CTAGC|TAGCTA|GCTAG|CTAGCT|AGCTA|GCTAGC|**
7&8: **TAGCTAG|CTAGCTAG|CTAGCTA|GCTAGCTA|GCTAGCT|AGCTAGCT|**
AGCTAGC|TAGCTAGC|
9&10: **TAGCTAGCT|AGCTAGCTAG|CTAGCTAGC|TAGCTAGCTA|GCTAGCTAG|**
CTAGCTAGCT|AGCTAGCTA|GCTAGCTAGC

If any other word sizes are tried, the new-size word would have to replace two or more smaller words to fit into the 220 letters. Since that would make fewer words, 40 words is the maximum.

2/4/14. We define the number s as

$$s = \sum_{i=1}^{\infty} \frac{1}{10^i - 1} = \frac{1}{9} + \frac{1}{99} + \frac{1}{999} + \frac{1}{9999} + \ldots = 0.12232424\ldots .$$

We can determine the nth digit right of the decimal point of s without summing the entire infinite series because after summing the first n terms of the series, the rest of the series sums to less than $2/10^{n+1}$. Determine the smallest prime number p for which the pth digit right of the decimal point of s is greater than 2. Justify your answer.

Comments: Igor Zhinitsky of New York, one of the contestants in the USAMTS, submitted the original version of this problem. We are thankful for his contribution and for his enthusiasm to participate in the USAMTS in more than one way. However, in the future, we are going to delay using any student-submitted problems until after the student has graduated from high school, in order to avoid a contestant submitting a solution to a problem that he or she wrote, as happened with Mr. Zhinitsky. However, this does give me an opportunity to present the problem writer's own solution as an official solution.

Solution 1 to 2/4/14 by Igor Zhinitsky (12/NY):

The answer to this problem is 47.

All the terms of the series have only 1's and 0's in all the digits after the decimal place. The nth digit of s can, thus, be found by calculating the number of terms that have a 1 in the nth place. 1/9 has a 1 in every place; thus, every place number divisible by 1 has a 1 from that term. 1/99 has a 1 in every other place; thus, every place number divisible by 2 has a 1 from that term. 1/999 has a 1 in every third place; thus, every place number divisible by 3 has a 1 from that term. And so forth.

Summing these values, we see that the nth digit is the number of factors of n, disregarding for the moment the possibility of 10 or more factors. More precisely, the nth digit $D(n)$ in general is exactly equal to

$$D(n) = \left(F(n) + \left\lfloor \frac{F(n+1) + \left\lfloor \frac{F(n+2) + \lfloor F(n+3) + \ldots \rfloor}{10} \right\rfloor}{10} \right\rfloor \right) \bmod 10$$

where $F(n)$ denotes the number of factors of n and $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

Clearly, the number of factors of any prime number is two; thus, the pth digit will be 2 unless the greatest integer part of the formula for $D(n)$ is non-zero, which will happen for the first time when $p+1$ has 10 or more factors. 48 is the first number that has 10 or more factors; thus, the 47th digit will be $F(47) + \left\lfloor \frac{10+0}{10} \right\rfloor$, which comes out to 3, because 47 is prime. Thus, 47 is the solution.

Solution 2 to 2/4/14 by Hilary Palevsky (12/PA):

When written in decimal form,

$$\frac{1}{9} = 0.111111\ldots = 0.\overline{1}$$

$$\frac{1}{99} = 0.010101\ldots = 0.\overline{01}$$

$$\frac{1}{999} = 0.001001\ldots = 0.\overline{001}$$

and so on. This can be generalized that $\frac{1}{10^i - 1}$ yields a number such that every nth digit right of the decimal point is 1 for all positive integers n divisible by i and all other digits are 0's.

From this, it can be seen that in the number s such that $s = \sum_{i=1}^{\infty} \frac{1}{10^i - 1}$, the value of the nth digit to the right of the decimal point is primarily determined by the number of factors of n. The sixth digit to the right of the decimal point, for example, is 4, because 6 has four factors: 1, 2, 3, and 6.

Thus, it would seem that when n is prime, the nth digit could not be greater than 2, since by the definition of a prime number, there cannot be more than two factors (1 and n). The exception, however, occurs when $n + 1$ has 10 or more factors, which would carry the digit from the tens place and add it to the digit 2 from the number n's own factors.

The smallest integer with 10 or more factors is 48, which has exactly ten factors. Since the number before 48, 47, is a prime number, the 47th digit to the right of the decimal point will be $2 + 1 = 3$.

Hence, 47 is the smallest prime number p for which the pth digit right of the decimal point of s is greater than 2.

3/4/14. Find the real-numbered solution to the equation below and demonstrate that it is unique.

$$\frac{36}{\sqrt{x}} + \frac{9}{\sqrt{y}} = 42 - 9\sqrt{x} - \sqrt{y}$$

Comment: This problem was dug up from deep in Prof. George Berzsenyi's archives, which means we have no-one to blame but ourselves for the flaw in it. There are two legitimate ways to interpret the problem, and one of those interpretations has five solutions instead of one. So we graded the problem as if it were the participant's choice out of the following two problems:

3/4/14 version (a). Working solely in the real numbers, find the solution to the equation below and demonstrate that it is unique.

$$\frac{36}{\sqrt{x}} + \frac{9}{\sqrt{y}} = 42 - 9\sqrt{x} - \sqrt{y}$$

(In this version, the square root of a negative number is undefined.)

3/4/14 version (b). Working in the complex numbers, find all solutions to the equation below in which both *x* and *y* are real numbers.

$$\frac{36}{\sqrt{x}} + \frac{9}{\sqrt{y}} = 42 - 9\sqrt{x} - \sqrt{y}$$

(In this version, the square root of a negative number n is $(\sqrt{|n|})i$ where $i = \sqrt{-1}$.)

We apologize for the confusion that the potential four additional solutions caused. However, we are elated that some of the USAMTS participants were insightful enough to find those four additional solutions despite the contradicting statement about uniqueness.

Solution to 3/4/14 version (b) by Andrew Altheimer (10/NC):

First, get all the *x*'s on one side and the *y*'s on the other:

$$\frac{36}{\sqrt{x}} + 9\sqrt{x} = 42 - \frac{9}{\sqrt{y}} - \sqrt{y} \tag{1}$$

Pull out common factors on each side, including the $1/(\sqrt{x})$ and $1/(\sqrt{y})$, and complete the squares:

$$\frac{9}{\sqrt{x}}((\sqrt{x})^2 + 4) = \frac{-1}{\sqrt{y}}((\sqrt{y})^2 - 42\sqrt{y} + 9)$$

$$\frac{9}{\sqrt{x}}((\sqrt{x})^2 - 4\sqrt{x} + 4) + \frac{9}{\sqrt{x}}(4\sqrt{x}) = \frac{-1}{\sqrt{y}}((\sqrt{y})^2 - 6\sqrt{y} + 9) + \frac{-1}{\sqrt{y}}(-36\sqrt{y})$$

$$\frac{9}{\sqrt{x}}((\sqrt{x})^2 - 4\sqrt{x} + 4) + 36 = \frac{-1}{\sqrt{y}}((\sqrt{y})^2 - 6\sqrt{y} + 9) + 36$$

$$\frac{9}{\sqrt{x}}((\sqrt{x})^2 - 4\sqrt{x} + 4) = \frac{-1}{\sqrt{y}}((\sqrt{y})^2 - 6\sqrt{y} + 9)$$

$$\frac{9}{\sqrt{x}}(\sqrt{x}-2)^2 = \frac{-1}{\sqrt{y}}(\sqrt{y}-3)^2$$

If *x* and *y* are positive so that their square roots are real numbers, then the left side would be greater than or equal to zero while the right hand side would be less than or equal to zero. Since the only overlap is zero, the equation can be solved by setting both sides equal to zero. This gives $x = 4$ and $y = 9$.

Although this finds a solution to the problem, it does not prove that no other solutions exist, because we have not examined the case where *x* or *y* could be negative. (Neither is zero due to the division by $\sqrt{x}$ and $\sqrt{y}$.)

Both *x* and *y* cannot both be negative, because this would leave only one real term in the original equation, the 42, with no other nonzero real term to match it on the other side.

If *x* is negative while *y* is positive, then we have two imaginary terms, both on the left side in equation (1) above. The only situation where this left side can match the real number on the right side is when both imaginary terms cancel each other out: $\frac{36}{\sqrt{x}} + 9\sqrt{x} = 0$. That gives $x = -4$. Plugging in $x = -4$ into equation (1) gives $42 - \frac{9}{\sqrt{y}} - \sqrt{y} = 0$. Simplify it as $(\sqrt{y})^2 - 42\sqrt{y} + 9$ = 0 and use the quadratic formula to find $\sqrt{y} = 21 \pm 12\sqrt{3}$. Squaring that gives $y = 873 \pm 504\sqrt{3}$.

If *y* is negative while *x* is positive, then we again have two imaginary terms and again need them to cancel out: $-\frac{9}{\sqrt{y}} - \sqrt{y} = 0$. That gives $y = -9$. Plugging this back into equation (1) gives $\frac{36}{\sqrt{x}} + 9\sqrt{x} = 42$. Simplify it as $9(\sqrt{x})^2 - 42\sqrt{x} + 36 = 0$ and use the quadratic formula to find $\sqrt{x} = \frac{7 \pm \sqrt{13}}{3}$. Squaring that gives $x = \frac{62 \pm 14\sqrt{13}}{9}$.

This gives five solutions to the equation: $(4, 9)$, $(-4, 873 \pm 504\sqrt{3})$, and $\left(\frac{62 \pm 14\sqrt{13}}{9}, -9\right)$. Plugging these numbers back into the original equation then proves that all these are valid real-number solutions to the equation. Although four of these solutions will form imaginary numbers, these numbers cancel out and since the solutions themselves are real, they should be considered "real-numbered solutions."

Solution 1 to 3/4/14 version (a) by Kristina Simmons (12/WI):

The given equation is rearranged to be $9\left(\sqrt{x} + \dfrac{4}{\sqrt{x}}\right) + \left(\sqrt{y} + \dfrac{9}{\sqrt{y}}\right) = 42.$

Substituting $m = \sqrt{x} + \dfrac{4}{\sqrt{x}}$ and $n = \sqrt{y} + \dfrac{9}{\sqrt{y}}$ gives $9m + n = 42$. Because $\sqrt{x} \ge 0$ and $\sqrt{y} \ge 0$ by definition, and $\sqrt{x} \ne 0$ and $\sqrt{y} \ne 0$ to avoid division by 0, m and n must both be positive. Now the task is to find the solution to $9m + n = 42$ for which $\sqrt{x}$ and $\sqrt{y}$ are real numbers.

The equation $m = \sqrt{x} + \dfrac{4}{\sqrt{x}}$, or equivalently $x - m\sqrt{x} + 4 = 0$, is quadratic in $\sqrt{x}$ with solutions $\sqrt{x} = \dfrac{m \pm \sqrt{m^2 - 16}}{2}$. For there to be at least one real solution, we need $m^2 - 16 \ge 0$, so $m \ge 4$, because m must be positive. Similarly, $n = \sqrt{y} + \dfrac{9}{\sqrt{y}}$ gives $\sqrt{y} = \dfrac{n \pm \sqrt{n^2 - 36}}{2}$, so $n \ge 6$. In other words, there is a real solution to the original equation only if $m \ge 4$ and $n \ge 6$.

However, if $m \ge 4$ and $n \ge 6$, then $9m + n \ge 42$. Since $9m + n = 42$, this forces $m = 4$ and $n = 6$. Now we can find x and y. Substituting for m, we get $\sqrt{x} = \dfrac{4 \pm \sqrt{4^2 - 16}}{2} = 2$, and substituting for n, we get $\sqrt{y} = \dfrac{6 \pm \sqrt{6^2 - 36}}{2} = 3$. Therefore, the unique solution to the equation is $(x, y) = (4, 9)$.

Solution 2 to 3/4/14 version (a) by Joanne Kong (11/NY):

This problem can be simplified by comparing arithmetic means with geometric means. For two positive numbers a and b, the arithmetic mean of the numbers is greater the geometric mean, or equal if the numbers are equal. $\frac{a+b}{2} \geq \sqrt{ab}$, so $a + b \geq 2\sqrt{ab}$.

The equation to be solved can be arranged into $\frac{36}{\sqrt{x}} + 9\sqrt{x} + \frac{9}{\sqrt{y}} + \sqrt{y} = 42$, and the sums $\frac{36}{\sqrt{x}} + 9\sqrt{x}$ and $\frac{9}{\sqrt{y}} + \sqrt{y}$ can be both viewed as twice an arithmetic mean. Every term is positive, since square roots are nonnegative and these square roots are not zero. So

$$\frac{36}{\sqrt{x}} + 9\sqrt{x} \geq 2\sqrt{\left(\frac{36}{\sqrt{x}}\right)(9\sqrt{x})} = 2\sqrt{324} = 36$$

and

$$\frac{9}{\sqrt{y}} + \sqrt{y} \geq 2\sqrt{\left(\frac{9}{\sqrt{y}}\right)(\sqrt{y})} = 2\sqrt{9} = 6.$$

The sum of the minimum values, 36 and 6, is 42, so equality is necessary. This means that the two terms inside each sum $\frac{36}{\sqrt{x}} + 9\sqrt{x}$ and $\frac{9}{\sqrt{y}} + \sqrt{y}$ are equal. $\frac{36}{\sqrt{x}} = 9\sqrt{x}$ gives $\frac{36}{9} = \sqrt{x} \cdot \sqrt{x} = x$, so $x = 4$. $\frac{9}{\sqrt{y}} = \sqrt{y}$ gives $9 = \sqrt{y} \cdot \sqrt{y} = y$, so $y = 9$.

The solution is $x = 4$ and $y = 9$.

Solution 3 to 3/4/14 version (a) by Daniel Walton (12/WA):

Start by substituting *a* and *b* for $\sqrt{x}$ and $\sqrt{y}$, respectively. $\frac{36}{\sqrt{x}} + \frac{9}{\sqrt{y}} = 42 - 9\sqrt{x} - \sqrt{y}$ becomes $\frac{36}{a} + \frac{9}{b} = 42 - 9a - b$. Now multiply both sides by *ab* and rearrange to solve for *b*.

$$36b + 9a = 42ab - 9a^2b - ab^2$$

$$ab^2 + 9a^2b - 42ab + 36b + 9a = 0$$

$$(a)b^2 + (9a^2 - 42a + 36)b + (9a) = 0$$

Use the quadratic formula and simplify.

$$b = \frac{-(9a^2 - 42a + 36) \pm \sqrt{(9a^2 - 42a + 36)^2 - 4(a)(9a)}}{2a}$$

$$b = \frac{-9a^2 + 42a - 36 \pm \sqrt{(81a^4 - 756a^3 + 2412a^2 - 3024a + 1296) - (36a^2)}}{2a}$$

$$b = \frac{-9a^2 + 42a - 36 \pm \sqrt{81a^4 - 756a^3 + 2376a^2 - 3024a + 1296}}{2a}$$

The discriminant $81a^4 - 756a^3 + 2376a^2 - 3024a + 1296$ factors as $3(a-2)^2(3a^2 - 16a + 12)$, so it is nonnegative whenever $a = 2$ or $a \le 0.903$ or $a \ge 4.43$. Remember that both *a* and *b* must be positive, since each is a real square root and is used as a denominator. So the bounds on *a* are $a = 2$ or $0 < a \le 0.903$ or $a \ge 4.43$.

Now examine the value of *b* for those values of *a*. For $0 < a \le 0.903$, $b < 0$. For $a = 2$, $b = 3$. For $a \ge 4.43$, $b < 0$. By elimination, *a* must be 2.

Does it work? $x = a^2 = 2^2 = 4$, $y = b^2 = 3^2 = 9$.

$$\frac{36}{\sqrt{4}} + \frac{9}{\sqrt{9}} \stackrel{?}{=} 42 - 9\sqrt{4} - \sqrt{9}$$

$$18 + 3 \stackrel{?}{=} 42 - 18 - 3$$

$$21 \stackrel{\checkmark}{=} 21 \qquad \text{Yes!}$$

Solution 4 to 3/4/14 version (a) by Ryan Hendrickson (12/MD):

Answer: $x = 4$, $y = 9$.

Proof of uniqueness:

$$\frac{36}{\sqrt{x}} + \frac{9}{\sqrt{y}} = 42 - 9\sqrt{x} - \sqrt{y}$$

$$\frac{36}{\sqrt{x}} + 9\sqrt{x} + \frac{9}{\sqrt{y}} + \sqrt{y} = 42$$

$$f(\sqrt{x}) + g(\sqrt{y}) = 42$$

where $f(z) = \frac{36}{z} + 9z$ and $g(z) = \frac{9}{z} + z$ for all real numbers z with $z > 0$. (That's all the domain we need, and it will make things easier.)

$\frac{df}{dz}(z) = -\frac{36}{z^2} + 9$ and $\frac{dg}{dz}(z) = -\frac{9}{z^2} + 1$. Fortunately, we restricted the domain of the functions to positive values of z to avoid the singularity at $z = 0$. We find the minimum values of $f(z)$ and $g(z)$ by setting their derivatives to zero. $\frac{df}{dz}(z) = 0$ gives $z = 2$, and $\frac{dg}{dz}(z) = 0$ gives $z = 3$. We take the second derivatives to verify that both solutions are local minima: $\frac{d^2 f}{(dz)^2}(2) = \frac{72}{2^3} = 9 > 0$ and $\frac{d^2 f}{(dz)^2}(3) = \frac{18}{3^3} = \frac{2}{3} > 0$—so the extrema points are minima.

At the boundaries of our domain, $\lim_{z \to 0+} f(z) = +\infty$, $\lim_{z \to +\infty} f(z) = +\infty$, $\lim_{z \to 0+} g(z) = +\infty$, and $\lim_{z \to +\infty} g(z) = +\infty$, so the local minima are absolute minima.

$f_{\min} = \frac{36}{2} + 9 \cdot 2 = 36$ and $g_{\min} = \frac{9}{3} + 3 = 6$, so the absolute minimum value of $f(\sqrt{x}) + g(\sqrt{y})$ is $36 + 6 = 42$. This is the required value, and it only happens when $\sqrt{x} = 2$ and $\sqrt{y} = 3$, which implies that $x = 4$, $y = 9$ is the only solution.

4/4/14. Two overlapping triangles could divide the plane into up to eight regions, and three overlapping triangles could divide the plane into up to twenty regions. Find, with proof, the maximum number of regions into which six overlapping triangles could divide the plane. Describe or draw an arrangement of six triangles that divides the plane into that many regions.

Comments: This problem was developed by Prof. George Berzsenyi, the founder of the USAMTS.

The graders of this problem requested that I explain the weaknesses of the greedy algorithm, because many submitted solutions relied on it. The greedy algorithm is a rule of thumb for making efficient constructions that are too complicated to construct in one fell swoop: first construct the best solution you can find for size 1; next, extend the size 1 result as efficiently as possible to make a size 2 result; continue this process, extending the size *i* result as efficiently as possible to construct size $i + 1$, until you reach the desired size.

The greedy algorithm does indeed find the correct maximum number of regions for problem 4/4/14. However, finding a result by the greedy algorithm does not count as a proof that that result is the best, because for some problems, the greedy algorithm fails to find the best solution, even when each step uses the provably best extension of the previous result. In those counterexamples, the best solution for size *i* does not contain the best solution for size $i - 1$ inside it, so extending the $(i - 1)$ st solution does not yield the *i*th solution.

For example, consider the problem of finding the smallest positive integer with exactly *i* factors. The answer for one factor is that 1 has one factor. By the greedy algorithm, to find a potential solution for two factors, we multiply 1 (the previous result) by 2 (the smallest prime) to find that 2 is the smallest integer with two factors, $\{1, 2\}$. By the greedy algorithm, to find a potential solution for three factors, we multiply 2 by 2 to find that 4 is the smallest integer with three factors, $\{1, 2, 4\}$. By the greedy algorithm, to find a potential solution for four factors, we multiply 4 by 2 to find that 8 is a small integer with four factors, $\{1, 2, 4, 8\}$. But it is not the smallest integer with four factors, because 6 is. The greedy algorithm failed.

The following solution by Piotr Wojciechowski uses the greedy algorithm, but he is careful to never demand that the solution for *n* triangles must contain the solution for $n - 1$ triangles, so his argument is rigorous. The second solution, by Connie Yee, also uses the greedy algorithm; in additon, she proves that the greedy algorithm does give the best solution. The third solution, by Greg Evans, does not use the greedy algorithm.

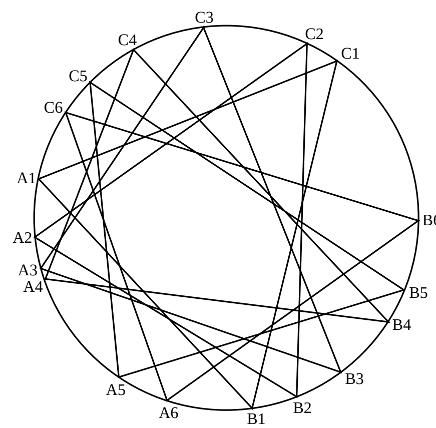
Solution 1 to 4/4/14 by Piotr Wojciechowski (9/WV):

Let R_n be the maximum number of regions into which *n* triangles can divide the plane. We will show by induction that $R_n \le 3n(n-1) + 2$.

Obviously, $R_1 = 2$. Suppose $T_1, T_2, \ldots, T_n$ are *n* triangles in the plane that divide the plane into R_n regions. Since a line can intersect a triangle at at most 2 points, a triangle (ignoring its interior) can intersect another triangle at at most 6 points. So a triangle can intersect $n - 1$ triangles at at most $6(n-1)$ points. Thus, triangle T_n intersects the other $n - 1$ triangles T_1, T_2, …, T_{n-1} in at most $6(n-1)$. If T_n intersects the other triangles at *k* points, then the perimeter of

T_n is divided into k intervals each of which divides a region created by the $n-1$ triangles into two regions. Thus, the number of regions into which the n triangles $T_1, T_2, \ldots, T_n$ divide the plane into is k larger than the number of regions into which the $n-1$ triangles $T_1, T_2, \ldots, T_{n-1}$ divide the plane. The number of regions created by the $n-1$ triangles is at most R_{n-1} and the value of k is at most $6(n-1)$. It follows that $R_n \le R_{n-1} + 6(n-1)$. By the induction hypothesis, $R_{n-1} \le 3(n-1)(n-2) + 2$. So $R_n \le 3(n-1)(n-2) + 2 + 6(n-1) = 3n(n-1) + 2$.
In particular, $R_6 \le 92$.

We will show that $R_6 = 92$, by showing that there is an arrangement of six triangles T_1, T_2, $\ldots$, T_6 on that plane that divide the plane into 92 regions. By the argument above, it is enough to arrange them so that any two triangles intersect at six points and no three triangles intersect at the same point.



The required arrangement can be obtained as follows. Take 18 points $A_1, A_2, \ldots, A_6, B_1, \ldots, B_6, C_1, \ldots, C_5, C_6$ on a circle, lying in that specificed order along the circle. Form the triangles $T_1 = \Delta A_1B_1C_1$ and $T_2 = \Delta A_2B_2C_2$. Clearly, T_1 and T_2 intersect at six points. If the segment $\overline{A_3B_3}$ goes through any of those intersection points, move point B_3 to anywhere else in the arc between B_2 and B_4 so that the segment $\overline{A_3B_3}$ does not go through any intersection points. This is possible since there are infinitely many possible positions for the point B_3 and only finitely many of those positions are bad. If any of the segments $\overline{A_3C_3}$ and $\overline{B_3C_3}$ go through an intersection point, then we can move point C_3 to a position so that the segments do not go through intersection points, as we did with point B_3. Once points B_3 and C_3 are in good positions, we form triangle $T_3 = \Delta A_3B_3C_3$. Repeat this process for B_4 and C_4 so that none of $\overline{A_4B_4}$, $\overline{A_4C_4}$, and $\overline{B_4C_4}$ go through any of the intersection points between T_1, T_2, and T_3, and form triangle $T_4 = \Delta A_4B_4C_4$. Repeat this process for B_5, C_5, B_6, and C_6 to form triangles $T_5 = \Delta A_5B_5C_5$ and $T_6 = \Delta A_6B_6C_6$. Because of the order of the points along the circle, any two triangles intersect at 6 points. Because we made sure to eliminate any triple intersections, no three triangles intersect at the same point.

Triangle T_2 is split into six parts by its points of intersection with triangle T_1, triangle T_3 is split into twelve parts by its points of intersection with triangles T_1 and T_2, and so on up to triangle T_6 split into 30 parts by its intersection with triangles T_1, T_2, T_3, T_4, and T_5. Since triangle T_1 alone divided the plane into two regions, the total number of regions is
$2 + 6 + 12 + 18 + 24 + 30 = 92$.

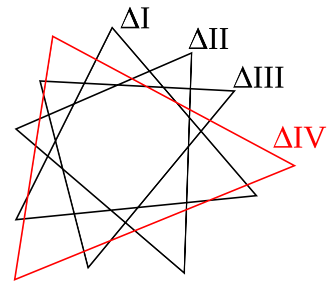
Solution 2 to 4/4/14 by Connie Yee (11/NY):

One triangle divides the plane into two regions.

For the best case with two triangles, each side of the second triangle cuts through each angle of the first triangle at two points. Each point of intersection corresponds to one new region. The three sides with two intersections each generate six more regions.

For the best case with three triangles, each side of the third triangle cuts through an angle of the first triangle and an angle of the second triangle for a total of four intersection points per side. Thus, the third triangle generates $3 \times 4 = 12$ more regions.

The fourth triangle can be set up so that each side of the fourth triangle cuts through an angle of the first triangle, an angle of the second triangle, and an angle of the third triangle, as shown in the diagram to the right. This gives a total of six intersection points per side, generating $3 \times 6 = 18$ more regions.

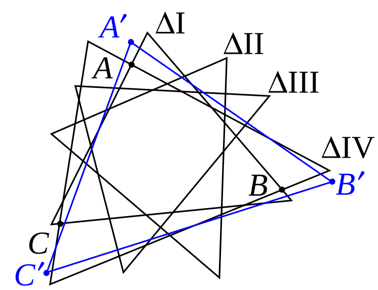


So long as we can send each side of the new triangle through an angle of each of the previous triangles, the pattern goes on. For each new intersection point, we get a new region. Each side can intersect a previous triangle at at most two points, by going through an angle. So the pattern give the maximum possible number of regions. The growth proceeds as follows:

Number of triangles	1	2	3	4	5	6	general n
Maximum number of regions	2	8 =2+3×2×1	20 =8+3×2×2	38 =20+3×2×3	62 =38+3×2×4	92 =62+3×2×5	R_n $= R_{n-1} +$ $6(n - 1)$

The recursive formula $R_n = R_{n-1} + 6(n-1)$ gives a direct formula $R_n = 3n(n-1) + 2$.

Can we always continue the pattern? Look at the four-triangle diagram again. Any side of triangle IV cuts through an angle of each of triangles I, II, and III, and that side and the next side of triangle IV make an angle of triangle IV. Label the first point where that side of triangle IV cuts through triangle I as A, the first point where the next side of triangle IV cuts through triangle I as B, and the first point of where the third side of triangle IV cuts through triangle I as C. Line $\overleftrightarrow{AB}$ cuts through an angle of every triangle, but line segment $\overline{AB}$ is barely too short to finish cutting through triangles I and IV, since its endpoints are on their perimeters. That happens to lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$, too. Points A, B, and C are next to the exterior region; therefore, we can find points A', B', and C' near them in the exterior region so that each side of triangle $A'B'C'$ cuts through an angle of every previous triangle. However, drawing a new side though an intersection point of previous triangles would not make a new intersection point and a new region. So we avoid drawing any new side through an old intersection.



We can make the sixth triangle by the same method.

Therefore, the maximum number of regions into which 6 overlapping triangles could divide a plane is 92.

Solution 3 to 4/4/14 by Greg Eden (10/MD):

To create the most regions in a plane with two overlapping triangles, the triangles should be placed so that they share an interior region but their vertices are rotated relative to each other so that each vertex of each triangle extends outside the other triangle. This creates 3+3 triangular regions from the corners of the triangles in addition to the region around the triangles and the polygonal region inside both triangles. This adds up to eight regions for two triangles, like the problem indicates.

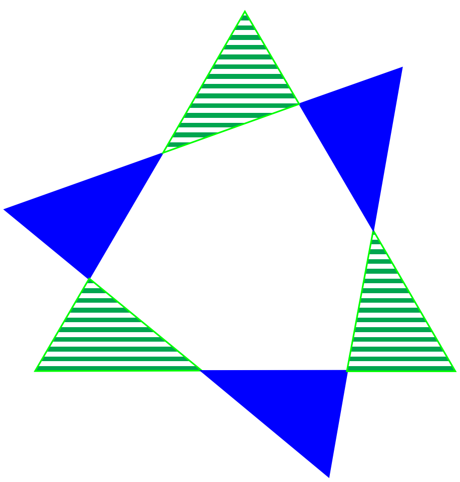
If there are x overlapping triangles, then there are

$\sum_{i=1}^{x-1} i = \frac{x(x-1)}{2}$ distinct pairs of triangles. Each distinct pair of triangle splits off six distinct triangular regions besides the common interior and exterior of each triangle. The interior and exterior regions are not created by the pairing of the triangles, but the six triangular regions are. Actually, with more than three overlapping triangles, these triangular regions lose their triangular shape, because pieces of them are claimed by the triangular regions split off to form the triangular regions of other pairs of triangles. But the number of formerly triangular regions is unaffected by losing pieces. This means that there are $6 \times \left(\frac{x(x-1)}{2}\right) = 3x(x-1)$ total formerly triangular regions formed by x overlapping triangles. When the original interior region and exterior region are added, there are a total of $r(x) = 3x(x-1)+2$ regions in the figure.

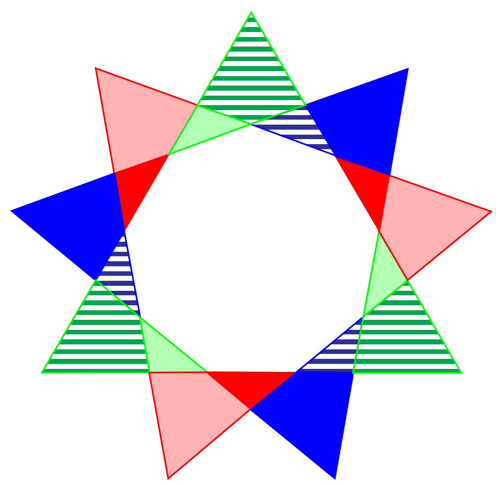
If a pair of overlapping triangles are positioned differently relative to each other, they cannot create more triangular regions than six, so $r(x)$ is the maximum.

For six overlapping triangles, this means we have $r(6) = 3 \cdot 6 \cdot 5 + 2 = 92$ regions.

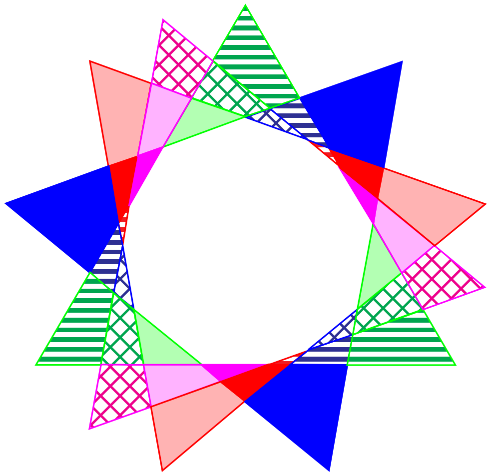
Colorful illustrations of how the triangular regions are split off are below.



2 triangles, 8 regions
Blue-green pair makes 3 solid blue regions and 3 striped green regions.

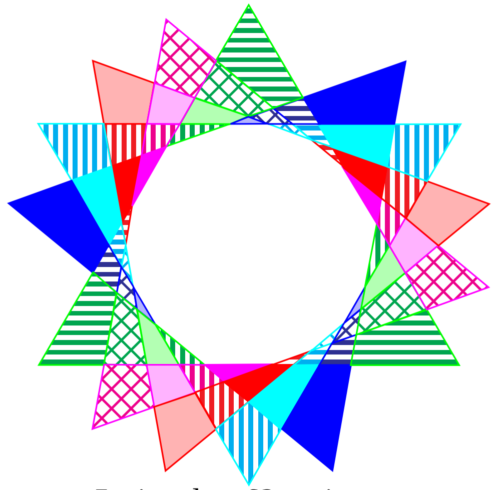


3 triangles, 20 regions
Red-blue pair makes 3 pale red regions and 3 striped blue regions. Red-green pair splits 3 solid blue regions into red and blue and 3 striped green regions into striped and pale.



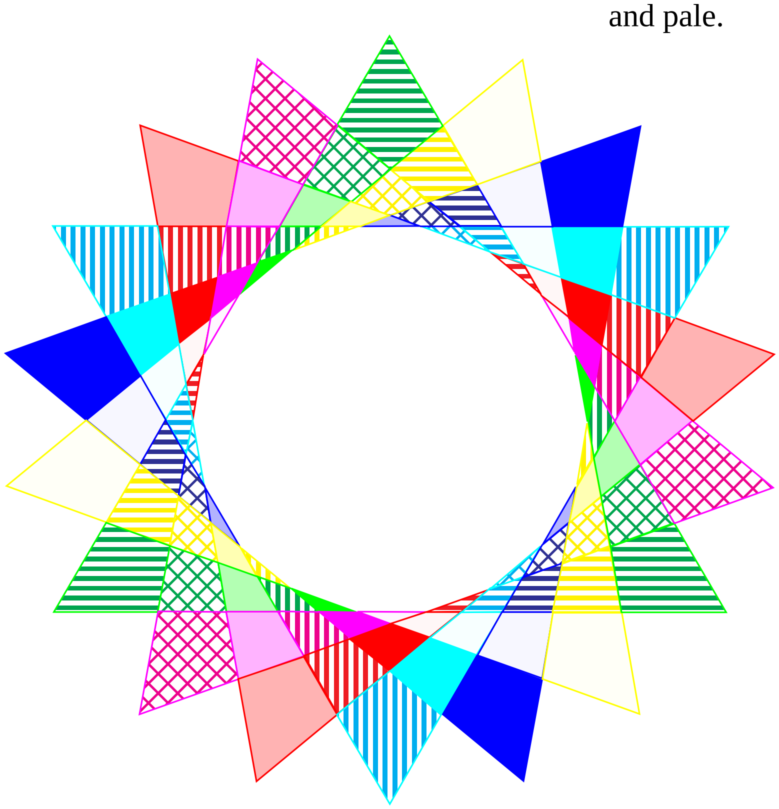
4 triangles, 38 regions

Magenta-red pair makes 3 hatched magenta regions and 3 striped red regions.
Magenta-blue pair splits 3 pale red regions into magenta and red and 3 striped blue regions into striped and hatched.
Magenta-green pair splits 3 solid red regions into magenta and red and 3 striped green regions into striped and hatched.



5 triangles, 62 regions

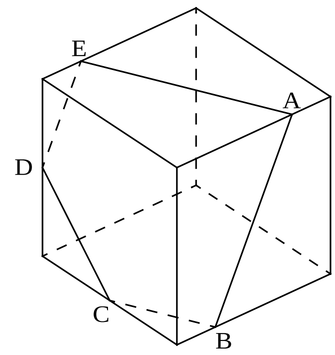
Cyan-blue pair makes 3 barred cyan regions and 3 pale blue regions.
Cyan-green pair splits 3 solid blue regions into cyan and blue and 3 pale green regions into barred and pale.
Cyan-magenta pair splits 3 striped blue regions into cyan and blue and 3 pale magenta regions into barred and pale.
Cyan-red pair splits 3 hatched blue regions into cyan and blue and 3 pale red regions into barred and pale.



6 triangles, 92 regions

Yellow-green pair makes 3 speckled yellow regions and 3 solid green regions.
Yellow-magenta pair splits 3 striped green regions into yellow and green and 3 solid magenta regions into speckled and solid.
Yellow-red pair splits 3 hatched green regions into yellow and green and 3 solid red regions into speckled and solid.
Yellow-cyan pair splits 3 pale green regions into yellow and green and 3 solid cyan regions into speckled and solid.
Yellow-blue pair splits 3 barred green regions into yellow and green and 3 solid blue regions into speckled and solid.

5/4/14. Prove that if the cross-section of a cube cut by a plane is a pentagon, as shown in the figure on the right, then there are two adjacent sides of the pentagon such that the sum of the lengths of those two sides is greater than the sum of the lengths of the other three sides. For ease of grading, please use the names of the points from the figure on the right in your solution.

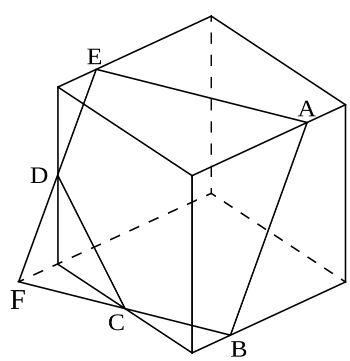


Comments: This problem was suggested by Professor Gregory Galperin of Eastern Illinois University. We are thankful for the many fine problems Prof. Galperin has sent the USAMTS over the years.

Solution 1 to 5/4/14 by Ari Officer (10/IL):

Extend the lines $\overleftrightarrow{ED}$ and $\overleftrightarrow{BC}$. Since both lines are on the plane cutting through the cube, they meet at a point F. The figure $ABFE$ is a parallelogram, since if two parallel planes, such as opposite sides of a cube, are intersected by a third plane, then the lines of intersection are parallel. Line segment $\overline{AE}$ is the same length as line segment $\overline{BF}$, and line segment $\overline{AB}$ is the same length as line segment $\overline{EF}$, because opposite sides of a parallelogram are of equal length. Thus, the sum of lengths AB and AE equals the sum of lengths BF and EF.

However, in the original pentagon, side $\overline{CD}$ cuts the corner of the parallelogram, removing F, which results in a decreased perimeter. The perimeter decreases in going from parallelogram to pentagon because the some of the lengths of two sides of a triangle, $\overline{CF}$ and $\overline{DF}$, is greater than the lenght of the third side, $\overline{CD}$. Since the sum of DE, CD, and BC is less than the sum of EF and BF, it is also less than the sum of AB and AE. Hence, the sum of the lengths of the two adjacent sides $\overline{AB}$ and $\overline{AE}$ is greather than the sum of the lengths of the other three sides $\overline{BC}$, $\overline{CD}$, and $\overline{DE}$.



Solution 2 for 5/4/14 by Judith Stanton (12/IN):

I start by creating a coordinate system. The vertex of the cube nearest points C and D is the origin, the edge containing C is on the positive x-axis, the edge containing D is on the positive z-axis, and the third edge out from the origin is on the positive y-axis. Let s be the length of an edge of the cube. Then I can write the coordinates of the vertices of the pentagon as:

$$A = (s, y_A, s), \quad B = (s, y_B, 0), \quad C = (x_C, 0, 0), \quad D = (0, 0, z_D), \quad E = (0, y_E, s).$$

Since the plane of the pentagon does not contain the origin, I can assume its equation is

$$mx - ny + lz = s. \tag{1}$$

and the reason of the subraction of the middle term will become obvious. Then substituting the coordinates for points A, B, C, D, and E into equation (1) gives

$$y_A = \frac{s(1-m-l)}{-n}, \quad y_B = \frac{s(1-m)}{-n}, \quad x_C = \frac{s}{m}, \quad z_D = \frac{s}{l}, \quad y_E = \frac{s(1-l)}{-n}.$$

Since x_C and z_D are between 0 and s, I see that $m, l > 1$. That means $1 - m - l < -1$, and since y_A is between 0 and s, it must be that $-n < 1 - m - l < -1$, which gives $n > 1$.

Let AB denote the distance between A and B, etc. Then

$$AB = \sqrt{(s-s)^2 + (y_A - y_B)^2 + (s-0)^2} = \sqrt{\frac{(sl)^2}{n^2} + s^2} = \frac{s}{n}\sqrt{l^2 + n^2}$$

$$BC = \sqrt{(s-x_C)^2 + (y_B - 0)^2 + (0-0)^2} = \sqrt{\frac{(s(m-1))^2}{m^2} + \frac{(s(m-1))^2}{n^2}} = \left(\frac{s}{n} - \frac{s}{mn}\right)\sqrt{m^2 + n^2}$$

$$CD = \sqrt{(x_C - 0)^2 + (0-0)^2 + (0 - z_D)^2} = \sqrt{\frac{s^2}{m^2} + \frac{s^2}{l^2}} = \frac{s}{lm}\sqrt{l^2 + m^2}$$

$$DE = \sqrt{(0-0)^2 + (0 - y_E)^2 + (z_D - s)^2} = \sqrt{\frac{(s(l-1))^2}{n^2} + \frac{(s(l-1))^2}{l^2}} = \left(\frac{s}{n} - \frac{s}{ln}\right)\sqrt{l^2 + n^2}$$

$$AE = \sqrt{(s-0)^2 + (y_A - y_E)^2 + (s-s)^2} = \sqrt{s^2 + \frac{(sm)^2}{n^2}} = \frac{s}{n}\sqrt{m^2 + n^2}.$$

So

$$AB + AE - BC - DE = \frac{s}{ln}\sqrt{l^2 + n^2} + \frac{s}{mn}\sqrt{m^2 + n^2} > 0$$

and

$$(AB + AE - BC - DE)^2 = \frac{s^2}{l^2} + \frac{s^2}{m^2} + \frac{2s^2}{n^2} + \frac{2s^2}{lmn^2}\sqrt{l^2 + n^2}\sqrt{m^2 + n^2}$$

$$> \frac{s^2}{l^2} + \frac{s^2}{m^2} = \left(\frac{s}{lm}\right)^2 (l^2 + m^2) = (CD)^2$$

Therefore, $AB + AE - BC - DE > CD$, so $AB + AE > BC + CD + DE$. Thus, the sum of the lengths of the adjacent sides $\overline{AB}$ and $\overline{AE}$ is greater than the sum of the lengths of the other three sides.