USA Mathematical Talent Search

PROBLEMS

Round 2 - Year 14 - Academic Year 2002-2003

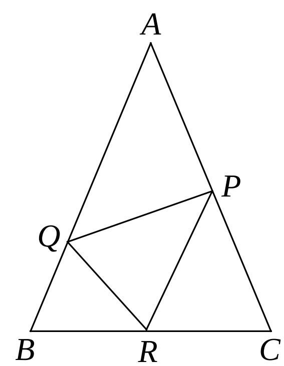
1/2/14. Each member of the sequence 112002, 11210, 1121, 117, 46, 34, … is obtained by adding five times the rightmost digit to the number formed by omitting that digit. Determine the billionth (10^9th) member of this sequence.

2/2/14. The integer 72 is the first of three consecutive integers 72, 73, and 74, that can each be expressed as the sum of the squares of two positive integers. The integers 72, 288, and 800 are the first three members of an infinite increasing sequence of integers with the above property. Find a function that generates the sequence and give the next three members.

3/2/14. An integer lattice point in the Cartesian plane is a point (x, y) where x and y are both integers. Suppose nine integer lattice points are chosen such that no three of them lie on the same line. Out of all 36 possible line segments between pairs of those nine points, some line segments may contain integer lattice points besides the original nine points. What is the minimum number of line segments that must contain an integer lattice point besides the original nine points? Prove your answer.

4/2/14. Let $f(n)$ be the number of ones that occur in the decimal representations of all the numbers from 1 to n. For example, this gives $f(8) = 1$, $f(9) = 1$, $f(10) = 2$, $f(11) = 4$, and $f(12) = 5$. Determine the value of $f(10^{100})$.

5/2/14. For an isosceles triangle ABC where $AB = AC$, it is possible to construct, using only compass and straightedge (see next page), an isosceles triangle PQR where $PQ = PR$ such that triangle PQR is similar to triangle ABC, point P is in the interior of line segment $\overline{AC}$, point Q is in the interior of line segment $\overline{AB}$, and point R is in the interior of line segment $\overline{BC}$. Describe one method of performing such a construction. Your method should work on every isosceles triangle ABC, except that you may choose an upper limit or lower limit on the size of angle BAC.



Complete, well-written solutions to at least two of the problems above, accompanied by a **Cover Sheet,** should be mailed to

USA Mathematical Talent Search
National Conference Services, Inc.
6440-C Dobbin Road
Columbia, MD 21045-4770

and **postmarked no later than November 24, 2002**. Each participant is expected to develop solutions without help from others. For the cover sheet and other details, see the USAMTS web site http://www.nsa.gov/programs/mepp/usamts.html.

Compass and Straightedge Constructions

Constructing geometric figures by compass and straightedge actually uses several idealized tools: a pen to draw with, paper to draw on, a compass to draw circles, a straightedge to draw lines, and a divider to copy distances. Classic geometers preferred these tools, as opposed to other tools such as the pins and string for drawing ellipses, because working with definite shapes and definite distances made proofs easier. Since some schools might not cover compass and straightedge constructions in their geometry curriculum, some USAMTS participants might benefit from a brief summary of some basic compass-and-straightedge constructions.

The following list covers some basic constructions. Combine them for more elaborate constructions. Points, lines, and circles can be used only if they are already drawn. A known distance is any distance between two points. A known angle is any angle between intersecting lines.

1. **Random Point.** Draw a point at a random location in the plane, on a line, on a line segment, on the circumference of a circle, or in the interior of any geometric figure.
2. **Line.** Draw a line through any two points. Draw a line through any point parallel to another line. Draw a line through any point perpendicular to another line. Extend a line segment into a line.
3. **Circle.** Draw a circle whose center is a point and whose radius is a known distance. Draw the circumscribed and inscribed circles of any triangle. Find the center point of a circle. Extend an arc into a circle.
4. **Intersection.** Find all points where two lines, two circles, or a line and a circle intersect.
5. **Distance.** Given a point on a line, draw another point on that line at a known distance from the first point in either direction. Can do the same with a line segment instead of a line, though it might be extend the line segment. Create any distance that is a rational or square root multiple of another known distance.
6. **Line Segment.** Draw a line segment between two points. Draw a copy of a line segment with one endpoint at any point and the copy rotated by any known angle.
7. **Angle.** Draw a line through a point on another line such that the angle between those lines copies any known angle. Draw an angle whose measure is equal to the sum or difference of the measures of any two known angles.
8. **Arc.** Draw an arc on a circle, starting at any point on the circumference, whose angle measure is any known angle.
9. **Partitioning.** Divide a line segment into any number of segments whose lengths are equal or are proportional to any collection of known distances.
10. **Angle Bisection.** Draw a line through the intersection of two other lines that bisects the angle between them. Trisection is not possible with compass and straightedge, though it is possible to construct 30°, 60°, and 90° angles.
11. **Tangent.** Draw two lines through a point exterior to a circle that are tangent to the circle. Draw a line through a point on the circumference of a circle that is tangent to the circle. Draw the common tangent lines to two circles, provided that a circle is not contained inside the other.
12. **Projection.** Copy any geometric figure build out of line segments and arcs, with its size rescaled by any ratio between known distances, its orientation rotated by any known angle, and one point of the copy at any desired point.