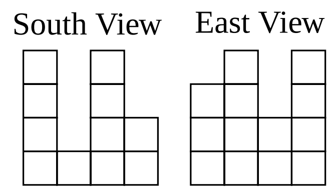
USA Mathematical Talent Search

PROBLEMS / SOLUTIONS / COMMENTS

Round 1 - Year 14 - Academic Year 2002-2003

1/1/14. Some unit cubes are stacked atop a flat 4 by 4 square. The figures show views of the stacks from two different sides. Find the maximum and minimum number of cubes that could be in the stacks. Also give top views of a maximum arrangement and a minimum arrangement with each stack marked with its height.



Comment: This problem was inspired by Problem 1097 in Sándor Róka's excellent collection of 2000 problems in elementary mathematics.

Solution 1 for 1/1/14 by Ameya Velingker (9/PA):

First, assign a coordinate system to the unit squares inside the 4 by 4 square as shown in figure 1.1. Let (i, j) denote the square that is in the i^{th} row and j^{th} column. Also let $S(i, j)$ denote the height of the stack on square (i, j).

(1,1)	(1,2)	(1,3)	(1,4)
(2,1)	(2,2)	(2,3)	(2,4)
(3,1)	(3,2)	(3,3)	(3,4)
(4,1)	(4,2)	(4,3)	(4,4)

Figure 1.1

Let us first find the maximal arrangement of cubes that will give the appropriate views. From looking at the south view, we can deduce that

$$S(k, 1) \le 4 \text{ for } 1 \le k \le 4$$
$$S(k, 2) \le 1 \text{ for } 1 \le k \le 4$$
$$S(k, 3) \le 4 \text{ for } 1 \le k \le 4$$
$$S(k, 4) \le 2 \text{ for } 1 \le k \le 4$$

By looking at the east view, we can also deduce that

$$S(1, k) \le 3 \text{ for } 1 \le k \le 4$$
$$S(2, k) \le 4 \text{ for } 1 \le k \le 4$$
$$S(3, k) \le 2 \text{ for } 1 \le k \le 4$$
$$S(4, k) \le 4 \text{ for } 1 \le k \le 4$$

For each square (i, j), we can choose the largest value for $S(i, j)$ that satisfies the above inequalities. This gives us the maximal arrangement with top view shown in figure 1.2. Each square in the figure contains the height of the stack on top of it. The total number of cubes used in this maximal configuration is 38.

4	1	4	2
2	1	2	2
4	1	4	2
3	1	3	2

Figure 1.2

Now let us find a minimal arrangement. Looking at the east view, we have

$$\max(S(1, 1), S(1, 2), S(1, 3), S(1, 4)) = 3$$
$$\max(S(2, 1), S(2, 2), S(2, 3), S(2, 4)) = 4$$
$$\max(S(3, 1), S(3, 2), S(3, 3), S(3, 4)) = 2$$
$$\max(S(4, 1), S(4, 2), S(4, 3), S(4, 4)) = 4$$

So, the number of cubes needed is at least 13. However, looking at the stack of height 1 in the south view, there is at least one value of k such that $S(k, 2) = 1$. Therefore, at least 14 cubes are needed. Figure 1.3 shows the top view of a working configuration using 14 cubes. Hence, this is a minimal configuration.

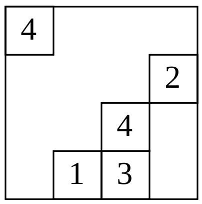
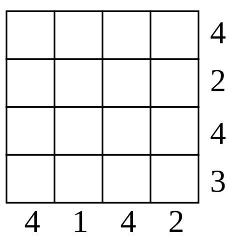


Figure 1.3

Solution 2 for 1/1/14 by James Albrecht (11/IL):

To prove the mximum and minimum for the number of blocks, first I will establish bounds and then I will find configurations that exist at the bounds.

Minimum: 14



According to the stacks seen from the east and the south, for the minimum we need at least:

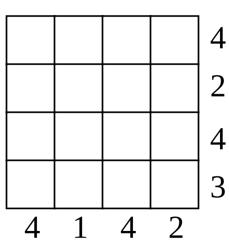
one stack of height 1,
one stack of height 2,
one stack of height 3, and
two stacks of height 4.

So min $\geq 1 + 2 + 3 + 4 + 4 = 14$.

There is a configuration with 14 cubes, so that is the minimum.

4	1	0	0
0	0	0	2
0	0	4	0
0	0	3	0

Maximum: 38



For the maximum, the south view lets us have 4's down the first column, 1's down the second column, 4's down the third column, and 2's down the fourth column.

But the east view limits the rows. In row 2 each entry must be no higher than 2. In row 4 each entry is no higher than 3. This leaves a configuration with 38 cubes.

4	1	4	2
2	1	2	2
4	1	4	2
3	1	3	2

2/1/13. Find four distinct positive integers, a, b, c, and d, such that each of the four sums $a+b+c$, $a+b+d$, $a+c+d$, and $b+c+d$ is the square of an integer. Show that infinitely many quadruples (a, b, c, d) with this property can be created.

Comment: This problem is based on Problem 5 of the 1993-94 Scottish Mathematical Challenge. The interested reader is directed to *Mathematical Challenges II*, published by the Scottish Mathematical Council in 1995.

Solution 1 for 2/1/14 by Matthew Walsh (12/NY):

One quadruple satisfying the conditions is (9, 198, 269, 522).

If we let $a = 3x^2 - 18x - 39$, $b = 3x^2 + 6$, $c = 3x^2 + 18x + 33$, and $d = 3x^2 + 36x + 42$ where x is some integer greater than 7 (so that a, b, c, and d are all positive), all of the four sums given in the problem will be perfect squares.

$$a+b+c = (3x^2-18x-39)+(3x^2+6)+(3x^2+18x+33) = 9x^2 = (3x)^2$$
$$a+b+d = (3x^2-18x-39)+(3x^2+6)+(3x^2+36x+42) = 9x^2+18x+9 = (3x+3)^2$$
$$a+c+d = (3x^2-18x-39)+(3x^2+18x+33)+(3x^2+36x+42) = 9x^2+36x+36 = (3x+6)^2$$
$$b+c+d = (3x^2+6)+(3x^2+18x+33)+(3x^2+36x+42) = 9x^2+54x+81 = (3x+9)^2$$

Because an infinite number of values for x exists, there also exists an infinite number of quadruples (a, b, c, d) such that each of the four sums $a+b+c$, $a+b+d$, $a+c+d$, and $b+c+d$ is a perfect square.

Solution 2 for 2/1/14 by Rohit Dewan (9/MD):

Let w, x, y, and z be the squares $a+b+c$, $a+b+d$, $a+c+d$, and $b+c+d$. So

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

Using a calculator, I found the inverse of $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ to be $\begin{bmatrix} 1/3 & 1/3 & 1/3 & -2/3 \\ 1/3 & 1/3 & -2/3 & 1/3 \\ 1/3 & -2/3 & 1/3 & 1/3 \\ -2/3 & 1/3 & 1/3 & 1/3 \end{bmatrix}$. Therefore,

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 & 1/3 & -2/3 \\ 1/3 & 1/3 & -2/3 & 1/3 \\ 1/3 & -2/3 & 1/3 & 1/3 \\ -2/3 & 1/3 & 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

which means

$$\begin{aligned} a &= (1/3)w + (1/3)x + (1/3)y - (2/3)z \\ b &= (1/3)w + (1/3)x - (2/3)y + (1/3)z \\ c &= (1/3)w - (2/3)x + (1/3)y + (1/3)z \\ d &= -(2/3)w + (1/3)x + (1/3)y + (1/3)z \end{aligned}$$

If *w*, *x*, *y*, and *z* are divisible by 3, then *a*, *b*, *c*, and *d* would be integers. I used 48^2 as *w*, 51^2 as *x*, 54^2 as *y*, and 57^2 as *z*. $48^2 = 2304$, $51^2 = 2601$, $54^2 = 2916$, and $57^2 = 3249$.

$$\begin{aligned} 441 &= (1/3)(2304) + (1/3)(2601) + (1/3)(2916) - (2/3)(3249) \\ 774 &= (1/3)(2304) + (1/3)(2601) - (2/3)(2916) + (1/3)(3249) \\ 1089 &= (1/3)(2304) - (2/3)(2601) + (1/3)(2916) + (1/3)(3249) \\ 1386 &= -(2/3)(2304) + (1/3)(2601) + (1/3)(2916) + (1/3)(3249) \end{aligned}$$

So (441, 774, 1089, 1386) is a working example.

If I were to multiply the 2304, 2601, 2916, and 3249 by any square, the new *a*, *b*, *c*, and *d* would be four different distinct positive integers. Therefore, infinitely many quadruples (a, b, c, d) can be created.

Comment from the solutions editor: Rohit Dewan's method for creating an infinite number of quadruples is simple, but the equations that are deduced are very powerful for creating quadruples. With a little rearrangement, we get

$$a = \frac{w + x + y + z}{3} - z$$

$$b = \frac{w + x + y + z}{3} - y$$

$$c = \frac{w + x + y + z}{3} - x$$

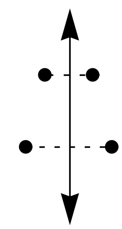
$$d = \frac{w + x + y + z}{3} - w$$

So in order for *a*, *b*, *c*, and *d* to be integers, we need only have the sum of the squares, $w + x + y + z$ divisible by 3 rather than having each individual square divisible by 3. With two more requirements that the squares *w*, *x*, *y*, and *z* be distinct, so that *a*, *b*, *c*, and *d* will be distinct, and that the largest of the squares be smaller than half the sum of the other three squares, so that the smallest of *a*, *b*, *c*, and *d* will be positive, these equations can generate all possible quadruples (a, b, c, d).

By using 8^2 as *w*, 9^2 as *x*, 10^2 as *y*, and 11^2 as *z* (note that $64 + 81 + 100 + 121 = 3 \times 122$), we get the smallest quadruple $(1, 22, 41, 58)$.

3/1/13. For a set of points in a plane, we construct the perpendicular bisectors of the line segments connecting every pair of those points and we count the number of points in which these perpendicular bisectors intersect each other. If we start with twelve points, the maximum possible number of intersection points is 1705. What is the maximum possible number of intersection points if we start with thirteen points?

Comment: This problem was found in an unpublished manuscript by János Pataki, a noted Hungarian mathematician.



We wish to apologize for a flaw in the statement of the problem. As Agustya Mehta's solution mentions below, it is possible to have an infinite number of points of intersection between the perpendicular bisectors, starting with as few as four points, if two line segments sharing the same perpendicular bisector are viewed as generating two superimposed perpendicular bisectors. We said that starting with twelve points gave 1705 intersections, so we felt that we had implicitly eliminated infinite intersections as a solution. However, we should have explicitly eliminated the infinite case by asking for the intersections of distinct perpendicular bisectors, as the editted problem below does. We simply had not noticed that the infinite case was possible in our translation from the problem's original Hungarian wording.

3/1/14. (editted) For a set of points in a plane, we construct the perpendicular bisectors of the line segments connecting every pair of those points. We count the number of points in which distinct perpendicular bisectors constructed this way intersect each other. If we start with twelve points, the maximum possible number of intersection points is 1705. What is the maximum possible number of intersection points if we start with thirteen points?

Solution 1 for 3/1/14 by Agustya Mehta (11/OH):

We assume that no two bisectors are common to each other, as this would result in infinite points of intersection.

If we start with x points, we will have $\binom{x}{2} = \frac{x(x-1)}{2}$ connecting lines, and thus also $\frac{x(x-1)}{2}$ perpendicular bisectors.

If we have n oblique lines with no restrictions on their placement, the mximum number of points of intersection that they create is $\binom{n}{2} = \frac{n(n-1)}{2}$.

In our case, any three of the starting points form a triangle, and thus their perpendicular bisectors will all intersect at a single point. This means that, for every set of three starting points, we will have a single point of intersection instead of the three given by $\binom{3}{2}$.

Thus, we must subtract $2\binom{x}{3}$ from the total possible number of points of intersection given by the formula $\frac{n(n-1)}{2}$ where $n = \frac{x(x-1)}{2}$ to correct the count.

Therefore, the maximum number of points of intersection for the perpendicular bisectors of the $(x(x-1))/2$ lines is

$$\frac{\left(\frac{x(x-1)}{2}\right)\left(\frac{x(x-1)}{2}-1\right)}{2}-2\binom{x}{3} = \frac{x(x-1)(x^2-x-2)}{2\cdot 2\cdot 2}-2\left(\frac{x(x-1)(x-2)}{3\cdot 2\cdot 1}\right)$$

$$= \frac{x(x-1)(x-2)(x+1)}{8}-\frac{x(x-1)(x-2)}{3}$$

$$= x(x-1)(x-2)\left(\frac{x+1}{8}-\frac{1}{3}\right)$$

$$= \frac{x(x-1)(x-2)(3x-5)}{24}$$

So, when $x = 12$, $\frac{x(x-1)(x-2)(3x-5)}{24} = \frac{12(11)(10)(31)}{24} = 1705$

and when $x = 13$, $\frac{x(x-1)(x-2)(3x-5)}{24} = \frac{13(12)(11)(34)}{24} = 2431$.

Solution 2 for 3/1/14 by Joel Lewis (12/NY):

We will approach the problem generally, for *n* points in the plane.

Any three points determine three segments, and thus three perpendicular bisectors. The three perpendicular bisectors are concurrent at the circumcenter of the triangle determined by the three points. So for every three points, we have one intersection. Thus, for *n* points in the plane, there are $C(n, 3)$ points of intersection of this type.

Also, if you take two separate pairs of points that are not in some special position, the perpendicular bisectors of their segments must intersect in a point. There are $C(n, 2)$ ways to pick the first pair of points and $C(n-2, 2)$ ways to pick the second pair. Order doesn't matter, so we have $C(n, 2)\times C(n-2, 2)/2$ total points of intersection of this type.

So in general, we have a maximum of $C(n, 3)+C(n, 2)\times C(n-2, 2)/2$ points of intersection. For $n = 12$ this gives us $220+66\times 45/2 = 220+1485 = 1705$, and for $n = 13$, this gives $286+78\times 55/2 = 286+2145 = 2431$.

Solution 3 for 3/1/14 by Mitka (Dmitry) Vaintrob (8/OR):

The answer is $C(C(13, 2), 2)-2C(13, 3) = 2431$. In general, if there are *n* points, there are at most $C(C(n, 2), 2)-2C(n, 3)$ intersection points. This is because for every pair of points, P and Q, there is one bisector $B_{P,Q}$ and each two bisectors intersect in at most one point. So the total number of pairs of bisectors $C(k, 2)$, where k is the number of bisectors $C(n, 2)$, is an upper bound on the number of intersection points.

For every three points, P, Q, and R, the three bisectors $B_{P,Q}$, $B_{P,R}$, and $B_{Q,R}$ are either parallel (this happens when P, Q, and R are collinear) or concurrent, intersecting at the circumcenter of triangle PQR. So for the three pairs of bisectors $\{B_{P,Q}, B_{P,R}\}$, $\{B_{P,Q}, B_{Q,R}\}$, $\{B_{P,R}, B_{Q,R}\}$, we get at most one intersection point, So we have to subtract at least two pairs of bisectors for each triple of points. This gives an upper bound of $C(C(n, 2), 2) - 2C(n, 3)$ for the maximum number of intersection points.

Let us now show that this number of intersections can indeed be achieved.

We will show by induction on *n* that there exist *n* points in the plane such that any two distinct bisectors of pairs of those points intersect and the only time three bisectors are concurrent is when they are of the form $B_{P, Q}$, $B_{P, R}$, $B_{Q, R}$ for any three points P, Q, R, and no four or more bisectors are concurrent.

Let us suppose the hypothesis is true for $n-1$ points and P_1, P_2, …, P_{n-1} are a configuration of points for which this is true. We will show that it is possible to add another point P_n so that the formula for number of intersections still hold for *n* points.

Notice that $B_{P, Q}$ is parallel to $B_{R, S}$ only if $\overline{PQ}$ is parallel to $\overline{RS}$, so if $\overline{P_iP_n}$ is not parallel to $\overline{P_jP_k}$ for any $1 \le i, j, k < n$ with $j < k$ then the bisectors B_{P_i, P_n} and B_{P_j, P_k} will intersect. So if P_n does not lie on a certain collection of lines through the points P_i, then all the bisectors will intersect.

Also notice that for a point O, the bisector $B_{P, Q}$ passes through O only if the distance OP equals the distance OQ. So given any point O where two or more existing bisectors intersect, the bisector B_{P_i, P_n} will not pass throught O if OP_n does not equal OP_i for $1 \le i < n$. So if P_n does not lie on a certain collection of circles whose centers are the points of intersection for the configuration of $n-1$ points, then no new perpendicular bisector will be concurrent with two or more existing perpendicular bisectors.

There is just one more type of concurrence that we didn't consider: a concurrence with two or more new bisectors involving P_n, such as a concurrence between B_{P_i, P_n}, B_{P_j, P_n}, and B_{P_k, P_l}, for $1 \le i, j, k, l < n$ with $i < j$ and $k < l$. In the example here, we must notice that the point of intersection of B_{P_i, P_n} and B_{P_j, P_n} is also on the old bisector B_{P_i, P_j} since P_i, P_j, and P_n form a triangle. So if $\{i, j\} \ne \{k, l\}$, both B_{P_i, P_j} and B_{P_k, P_l} would pass through that point, so we would have the concurrence at the intersection of two old bisectors, which brings this example into a previous case. The only other example possible would be when B_{P_i, P_n}, B_{P_j, P_n}, and B_{P_k, P_n} concur, but that intersection point is also on B_{P_i, P_j}, B_{P_i, P_k}, and B_{P_j, P_k}, which is again that previous case.

So, as long as P_n is not is not on a certain finite set of circles nor on a certain finite set of lines, the points P_1, P_2, …, P_{n-1}, P_n satisfy all the conditions.

Our induction hypothesis is true when $n = 3$, for when we have three points P_1, P_2, and P_3 not all on one line, the three bisectors intersect at the circumcenter of the triangle $P_1P_2P_3$ and $C(C(3, 2), 2) - 2C(3, 3) = C(3, 2) - 2C(3, 3) = 3 - 2 \cdot 1 = 1$.

Therefore, the number of intersections of bisectors for the collection of points we constructed is equal to $C(C(n, 2), 2) - 2C(n, 3)$. Which shows that our upper bound is indeed the maximum number. When $n = 13$, this gives the answer $C(C(13, 2), 2) - 2C(13, 3) = 2431$.

4/1/13. A transposition of a vector is created by switching exactly two entries of the vector. For example, $(1, 5, 3, 4, 2, 6, 7)$ is a transposition of $(1, 2, 3, 4, 5, 6, 7)$. Find the vector X if $S = (0, 0, 1, 1, 0, 1, 1)$, $T = (0, 0, 1, 1, 1, 1, 0)$, $U = (1, 0, 1, 0, 1, 1, 0)$, and $V = (1, 1, 0, 1, 0, 1, 0)$ are all transpositions of *X*. Describe your method for finding *X*.

Comment: This problem was created by Drs. Gene Berg and George Berzsenyi on the basis of an article by Vladimer I. Levenshtein, published in the January 2001 issue of the *IEEE Transactions on Information Theory.*

Solution 1 for 4/1/13 by Brian Ho (12/IL):

To find the vector *X*, we want to look for a vector as similar to *S*, *T*, *U*, and *V* as possible, since vector transposition does not change the vector very much.

$S = (0, 0, 1, 1, 0, 1, 1)$

$T = (0, 0, 1, 1, 1, 1, 0)$

$U = (1, 0, 1, 0, 1, 1, 0)$

$V = (1, 1, 0, 1, 0, 1, 0)$

We can notice that positions 2 and 7 have three of the four vectors as 0. Positions 3, 4, and 6 have three or four vectors as 1. Based on the above reasoning, we can say that $X = (?, 0, 1, 1, ?, 1, 0)$.

If we compare this *X* with vector *V*, we notice that the entries at positions 2 and 3 are switched. The unknown entries of *X* must thus be the same as the corresponding entries in *V*, because only two entries are switched in a transposition. Thus $X = (1, 0, 1, 1, 0, 1, 0)$.

We can verify that this is a viable vector *X* by testing that it is a transposition of *S*, *T*, and *U* as well as of *V*. Switching positions 1 and 7 in *X* results in *S*, switching positions 1 and 5 in *X* results in *T*, switching positions 4 and 5 results in *U*, and switching positions 2 and 3 results in *V*. We have, therefore, found *X*.

Comment from the solutions editor: Brian Ho's nice solution uses a method mathematicians call majority vote. It could be improved by determining when majority vote would yield correct information about *X*. As it is, Mr. Ho had to verify his answer at the end to be rigorous.

If the writer of the problem had added a fifth transposition $W = (0, 1, 1, 1, 0, 1, 0)$ to the other four transpositions, majority vote would have given $(0, 0, 1, 1, 0, 1, 0)$ as the result, which is not *X* and has too many zeros. Common sense tells us that given too many zeros, the 3-to-2 votes for 0 are the suspect entries, so the result converts to $(?, 0, 1, 1, ?, 1, 0)$, which we can handle as Mr. Ho did. But it is wise to support common sense with rigorous mathematics.

As Mr. Ho argued, a transposition is not changed from *X* by much: it differs in two entries (we will ignore switching two identical entries, since that would yield *X* itself and testing shows that neither *S*, *T*, *U*, nor *V* is *X)*. So the four transpositions combined differ from *X* at exactly eight entries. The two columns at positions 1 and 5 must use up four of those different entries, two in each column. That leaves four other different entries. We have four columns with a 3-to-1 vote, so the minority values in those column must be the four other entries that don't match *X*. Thus, we have proven Mr. Ho's claim that $X = (?, 0, 1, 1, ?, 1, 0)$.

Solution 2 for 4/1/14 by Yuyin Chen (11/MI):

I will list every transposition of S, T, U, and V.

S	T	U	V
$(0, 0, 0, 1, 1, 1, 1)$	$(0, 0, 0, 1, 1, 1, 1)$	$(0, 0, 1, 0, 1, 1, 1)$	$(0, 1, 0, 1, 0, 1, 1)$
$(0, 0, 1, 0, 1, 1, 1)$	$(0, 0, 1, 0, 1, 1, 1)$	$T = (0, 0, 1, 1, 1, 1, 0)$	$(0, 1, 0, 1, 1, 1, 0)$
$S = (0, 0, 1, 1, 0, 1, 1)$	$S = (0, 0, 1, 1, 0, 1, 1)$	$(0, 1, 1, 0, 1, 1, 0)$	$(0, 1, 1, 1, 0, 1, 0)$
$(0, 0, 1, 1, 1, 0, 1)$	$(0, 0, 1, 1, 1, 0, 1)$	$(1, 0, 0, 0, 1, 1, 1)$	$(1, 0, 0, 1, 0, 1, 1)$
$T = (0, 0, 1, 1, 1, 1, 0)$	$T = (0, 0, 1, 1, 1, 1, 0)$	$(1, 0, 0, 1, 1, 1, 0)$	$(1, 0, 0, 1, 1, 1, 0)$
$(0, 1, 0, 1, 0, 1, 1)$	$(0, 1, 0, 1, 1, 1, 0)$	$(1, 0, 1, 0, 0, 1, 1)$	$(1, 0, 1, 1, 0, 1, 0)$
$(0, 1, 1, 0, 0, 1, 1)$	$(0, 1, 1, 0, 1, 1, 0)$	$(1, 0, 1, 0, 1, 0, 1)$	$(1, 1, 0, 0, 0, 1, 1)$
$(0, 1, 1, 1, 0, 0, 1)$	$(0, 1, 1, 1, 0, 1, 0)$	$U = (1, 0, 1, 0, 1, 1, 0)$	$(1, 1, 0, 0, 1, 1, 0)$
$(0, 1, 1, 1, 0, 1, 0)$	$(0, 1, 1, 1, 1, 0, 0)$	$(1, 0, 1, 1, 0, 1, 0)$	$(1, 1, 0, 1, 0, 0, 1)$
$(1, 0, 0, 1, 0, 1, 1)$	$(1, 0, 0, 1, 1, 1, 0)$	$(1, 0, 1, 1, 1, 0, 0)$	$V = (1, 1, 0, 1, 0, 1, 0)$
$(1, 0, 1, 0, 0, 1, 1)$	$U = (1, 0, 1, 0, 1, 1, 0)$	$(1, 1, 0, 0, 1, 1, 0)$	$(1, 1, 0, 1, 1, 0, 0)$
$(1, 0, 1, 1, 0, 0, 1)$	$(1, 0, 1, 1, 0, 1, 0)$	$(1, 1, 1, 0, 0, 1, 0)$	$(1, 1, 1, 0, 0, 1, 0)$
$(1, 0, 1, 1, 0, 1, 0)$	$(1, 0, 1, 1, 1, 0, 0)$	$(1, 1, 1, 0, 1, 0, 0)$	$(1, 1, 1, 1, 0, 0, 0)$

Now I will systematically search for any vector that is in all four lists. If a vector is in the S list and not in the T list, I do not have to search for it in the U list, and if a vector is in the S and T lists and not in the U list, I do not have to search for it in the V list. So 13 vectors in the S list reduce to 7 vectors in the T list, then to 3 vectors in the U list, and then to 1 vector in the V list, which is $(1, 0, 1, 1, 0, 1, 0)$. It is a transposition of all four vectors; therefore, all four vectors are transpositions of it. So our solution is $X = (1, 0, 1, 1, 0, 1, 0)$.

Comment from the solutions editor: I sorted Yuyin Chen's lists into lexicographical order, because such an ordering makes comparisons between lists easier.

Solution 3 to 4/1/14 by Michael Chmutov (12/OM):

Suppose the first entry of X is 0. Then in the vectors U and V it was changed by the transposition. In the vector U that 0 would have been switched with either the second, fourth, or seventh entry, which would make X either $(0, 1, 1, 0, 1, 1, 0)$, $(0, 0, 1, 1, 1, 1, 0)$, or $(0, 0, 1, 0, 1, 1, 1)$. But V is not a transposition of any of these. Consequently, the first entry of X is 1.

In the vector S that first entry of X was switched with either the third, fourth, sixth, or seventh entry, which makes X either $(1, 0, 0, 1, 0, 1, 1)$, $(1, 0, 1, 0, 0, 1, 1)$, $(1, 0, 1, 1, 0, 0, 1)$, or $(1, 0, 1, 1, 0, 1, 0)$. The only one of these for which T is a transposition is $(1, 0, 1, 1, 0, 1, 0)$. Checking it against U and V, they also turn out to be transpositions of it.

Answer: $X = (1, 0, 1, 1, 0, 1, 0)$.

Solution 4 for 4/1/14 by Michael Lieberman (12/PA):

First I will note that if two transpositions of X are different in four places, then two of those places must have been transposed in one and the other two in the other. Therefore, the remaining three places must all be the same as the original vector X. For example, U and V are different in positions 2, 3, 4, and 5, so their entries at positions 1, 6, and 7 must be the entries in X.

$$\begin{array}{rcl} U & = & (1, 0, 1, 0, 1, 1, 0) \\ V & = & (1, 1, 0, 1, 0, 1, 0) \\ \hline X & = & (1, ?, ?, ?, ?, 1, 0) \end{array}$$

Similarly, by comparing S with U and S with V the rest of X can be determined.

$$\begin{array}{rcl} S & = & (0, 0, 1, 1, 0, 1, 1) \\ U & = & (1, 0, 1, 0, 1, 1, 0) \\ \hline X & = & (?, 0, 1, ?, ?, 1, ?) \end{array} \qquad \begin{array}{rcl} S & = & (0, 0, 1, 1, 0, 1, 1) \\ V & = & (1, 1, 0, 1, 0, 1, 0) \\ \hline X & = & (?, ?, ?, 1, 0, 1, ?) \end{array}$$

$$\begin{array}{rcl} & & (1, ?, ?, ?, ?, 1, 0) \\ & & (?, 0, 1, ?, ?, 1, ?) \\ & & (?, ?, ?, 1, 0, 1, ?) \\ \cline{3-3} X & = & (1, 0, 1, 1, 0, 1, 0) \end{array}$$

Note that vector T is not even needed, but it can be used to check the correct answer. The following diagram shows which entries were switched.

$$\begin{array}{rcl} X & = & (1, 0, 1, 1, 0, 1, 0) \\ S & = & (\boxed{0}, 0, 1, 1, 0, 1, \boxed{1}) \\ T & = & (\boxed{0}, 0, 1, 1, \boxed{1}, 1, 0) \\ U & = & (1, 0, 1, \boxed{0}, \boxed{1}, 1, 0) \\ V & = & (1, \boxed{1}, \boxed{0}, 1, 0, 1, 0) \end{array}$$

Solution 5 for 4/1/14 by Brandon Levin (12/OH):

Let $X = (x_1, x_2, x_3, x_4, x_5, x_6, x_7)$. Observe that X must contain four 1's and three 0's since S, T, U, and V do. Also note that none of S, T, U, and V can be X.

Now the dot product $X \bullet X$ is 4, since there are four 1's. (A dot product of two vectors is the sum of all the pairwise products of entries in the same positions.) However, when one of the 1's is switched with a 0 to make a transposition, the dot product of X and the transposition will be less by 1, in other words, the dot product will be 3.

Therefore, the dot product of X with any of the transpositions S, T, U, and V will be 3. Also the dot product of known vector, such as S, with X will be the sum of the entries in X : the variables x_1, x_2, x_3, x_4, x_5, x_6, and x_7. So, each dot product of X with the transpositions S, T, U, and V will create an equation in the variables x_1, x_2, x_3, x_4, x_5, x_6, and x_7.

Therefore,

$$X \bullet S = x_3 + x_4 + x_6 + x_7 = 3$$
$$X \bullet T = x_3 + x_4 + x_5 + x_6 = 3$$
$$X \bullet U = x_1 + x_3 + x_5 + x_6 = 3$$
$$X \bullet V = x_1 + x_2 + x_4 + x_6 = 3$$

For solving the equations, we can use the extra information that the seven variables consist of four 1's and three 0's. So three of the variables in each equation above are 1 and the fourth variable is 0. Since x_6 is in every equation, if it were 0, then all the other variables would have to be 1, which cannot be true. So $x_6 = 1$.

Also, subtracting the first two equations from each other gives that $x_5 = x_7$ and subtracting the middle two equations from each other gives that $x_1 = x_4$. Substituting for x_4, x_6, and x_7 leaves two distinct equations:

$$x_1 + x_3 + x_5 = 2$$
$$2x_1 + x_2 = 2$$

That last equation can be solved with 0's and 1's only by $x_1 = 1$ and $x_2 = 0$.

We know three of the four 1's in X already: x_1, x_4, and x_6. There is only one 1 left and since $x_5 = x_7$, they cannot be 1. So x_2, x_5, and x_7 are 0, which leaves that $x_3 = 1$.

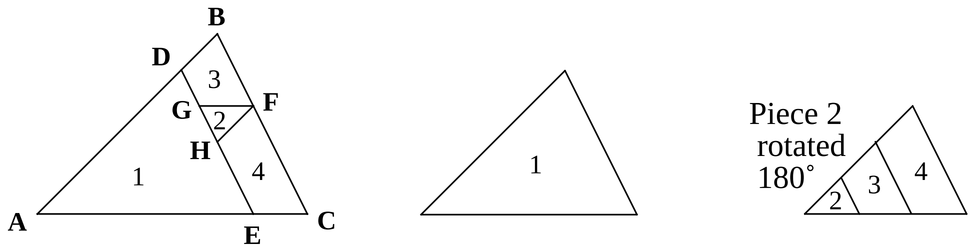
Finally, $X = (1, 0, 1, 1, 0, 1, 0)$.

Comment from the solutions editor: Brandon Levin could have used the all-ones vector $\vec{1} = (1, 1, 1, 1, 1, 1, 1)$ to create a fifth equation:

$$X \bullet \vec{1} = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 4$$

which represents the information that X contains exactly four 1's.

5/1/14. As illustrated below, we can dissect every triangle **ABC** into four pieces so that piece 1 is a triangle similar to the original triangle, while the other three pieces can be assembled into a triangle also similar to the original triangle. Determine the ratios of the sizes of the three triangles and verify that the construction works.



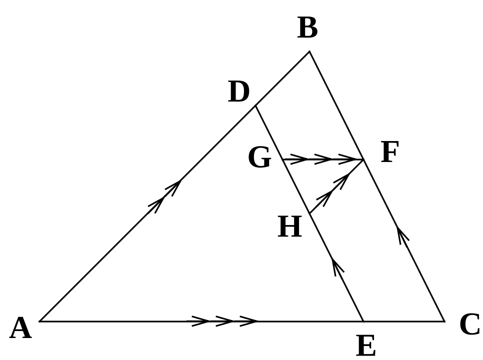
Comment: This problem was inspired by Problem 1365 in Sándor Róka's excellent collection of 2000 problems in elementary mathematics, which is also Problem 1113 in his earlier collection of only 1500 problems.

Some solutions showed that several people were unsure what the instruction "verify that the construction works" meant. It meant to show that the pieces fit together properly—by showing that sides that had to fit together had the same length and angles that had to fit together into a straight line were supplementary—and that the three triangles were similar.

We counted both the ratios of the lengths, 5 : 4 : 3, and the ratios of the areas, 25 : 16 : 9, as correct answers. "Size" is an ambiguous word, but the size of a polygon usually means the lengths of its sides.

Solution 1 for 5/1/14 by Kristen Kozak (11/WA):

The construction works if we make the dissection lines of triangle **ABC** parallel to its sides as shown below.



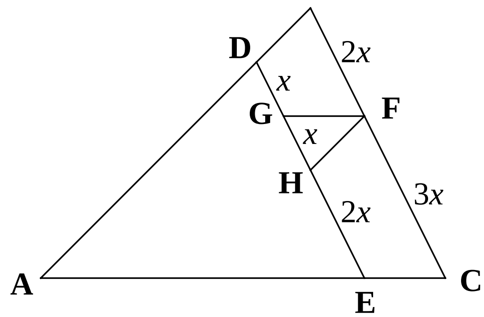
Making the lines parallel also allows us to make the following equalities

DB = **HF**	Opposite sides of parallelogram **DBFH**.
DH = **BF**	Opposite sides of parallelogram **DBFH**.
GF = **EC**	Opposite sides of parallelogram **GFCE**.
GE = **FC**	Opposite sides of parallelogram **GFCE**.

By looking at the assembled triangle, we obtain these equalities.

DG = **GH**	Aligned where pieces 2 and 3 touch on assembled triangle.
HE = **BF**	Aligned where pieces 3 and 4 touch on assembled triangle.

We find the ratio of the size of triangle **ADE** to the size of triangle **ABC** by finding the ratio of the length of segment $\overline{\mathbf{DE}}$ to the length of segment $\overline{\mathbf{BC}}$, because $\overline{\mathbf{DE}}$ and $\overline{\mathbf{BC}}$ are corresponding edges. First we define *x* to be the length of segment $\overline{\mathbf{DG}}$. Then we use the equalities above to find the lengths of the the other line segments on segments $\overline{\mathbf{DE}}$ and $\overline{\mathbf{BC}}$ in terms of *x*. These lengths are shown below.



We find the length of $\overline{\mathbf{DE}}$ by summing the lengths of $\overline{\mathbf{DG}}$, $\overline{\mathbf{GH}}$, and $\overline{\mathbf{HE}}$. We find the length of $\overline{\mathbf{BC}}$ by summing the lengths of $\overline{\mathbf{BF}}$ and $\overline{\mathbf{FC}}$. $\mathbf{DE} = 4x$. $\mathbf{BC} = 5x$. The ratio of the length of side $\overline{\mathbf{DE}}$ to the length of side $\overline{\mathbf{BC}}$ is $4x : 5x$, which is 4 : 5. We square the ration of the lengths to find the ratio of the area of triangle **ADE** to the area of triangle **ABC**. We get 16 : 25.

We find the ratio of the size of the assembled triangle to the size of triangle ABC by finding the ratio of the length of segment $\overline{\mathbf{FC}}$ to the length of segment $\overline{\mathbf{BC}}$, because $\overline{\mathbf{FC}}$ and $\overline{\mathbf{BC}}$ are corresponding edges. We know that the length of $\overline{\mathbf{FC}}$ is $3x$ and the length of $\overline{\mathbf{BC}}$ is $5x$, so the ratio of their lengths is 3 : 5. We square the ratio of the lengths to find the ratio of the areas of the two triangles. We get 9 : 25.

So, the ratio of the area of triangle **ABC** to the area of triangle **ADE** to the area of the assembled triangle is 25 : 16 : 9.

Solution 2 for 5/1/14 by Sam McVeety (11/MN):

In verifying that this construction works, I found a Euclidean method of creating the figure that clearly shows both the area ratios and verifies the construction. First, one must divide base $\overline{\mathbf{AC}}$ into five equal parts.[1] We now construct parallels to side $\overline{\mathbf{BC}}$ at the four points dividing those five parts of $\overline{\mathbf{AC}}$. Those parallel lines divide side $\overline{\mathbf{AB}}$ into five equal parts. Next, construct paral-

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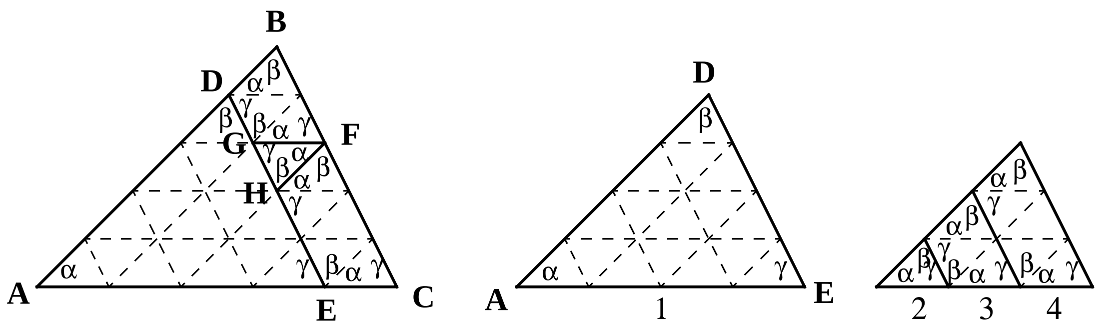
1. Using Euclid's method, we draw a line $\overleftrightarrow{\mathbf{AM}}$ where **M** is a point not on line $\overleftrightarrow{\mathbf{AC}}$ and create five adjacent equal segments on $\overleftrightarrow{\mathbf{AM}}$ beginning at **A**, labeling the endpoint of the last segment as **N**. We next draw line segment $\overline{\mathbf{CN}}$ and construct a parallel to $\overline{\mathbf{CN}}$ at every endpoint of the equal segments on $\overleftrightarrow{\mathbf{AM}}$. These parallel lines will divide line segment $\overline{\mathbf{AC}}$ into five equal parts.

lels to side $\overline{\mathbf{AC}}$ at the four points on $\overline{\mathbf{AB}}$ where the parallels to side $\overline{\mathbf{BC}}$ intersected. Those parallel lines divide side $\overline{\mathbf{BC}}$ into five equal parts. Finally, constuct parallels to side $\overline{\mathbf{AB}}$ from the four points on $\overline{\mathbf{AC}}$ where the parallels to side $\overline{\mathbf{AC}}$ intersected. Those parallel lines pass through the original four points of penta-section on side $\overline{\mathbf{AC}}$. We have now created 25 congruent triangles, which are similar to the original triangle **ABC** because the parallel lines cut equal angles at their intersections with parallel lines, and are congruent to each other because adjacent triangles share a matching side and triangles along each side of the large triangle **ABC** have sides that are from five equal parts of that side.



The divided triangle and its similar triangles, with important angles labeled

In the construction, line segment $\overline{\mathbf{DE}}$ lies over one of the parallels to $\overline{\mathbf{BC}}$, line segment $\overline{\mathbf{FG}}$ lies over one of the parallels to $\overline{\mathbf{AC}}$, and line segment $\overline{\mathbf{FH}}$ lies over one of the parallels to $\overline{\mathbf{AB}}$, as shown in the diagram above. Piece 1 of triangle **ABC** consists of 16 of the 25 tiny triangles, piece 2 is one of the 25 tiny triangles, piece 3 consists of three of the 25 tiny triangles, and piece 4 consists of five of the 25 tiny triangles. Since the tiny triangles fit together, pieces 2, 3, and 4 fit together to form a triangle made of nine of the tiny triangles.

Clearly, the ratio of the areas of triangle **ABC** to triangle **ADE** to the assembled triangle is 25 : 16 : 9.

Extending this construction to dividing triangles into n parts in similar fashion,[1] where n is even and greater than 2, yields the following ratio of areas:

$$\left(\frac{n^2-2n+2}{2}\right)^2 : \left(\frac{n^2-2n}{2}\right)^2 : (n-1)^2$$

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1. The triangle is divided so that one large part is similar to the original triangle, and the other $n-1$ pieces are a small triangle and several trapezoids of increasing size that are assembled into a triangle similar to the original. For example, $n = 6$ would be: