USA Mathematical Talent Search

PROBLEMS / SOLUTIONS / COMMENTS

Round 4 - Year 13 - Academic Year 2001-2002

solutions edited by Erin Schram

1/4/13. In a strange language there are only two letters, *a* and *b*, and it is postulated that the letter *a* is a word. Furthermore, all additional words are formed according to the following rules:

A. Given any word, a new word can be formed from it by adding a *b* at the righthand end.

B. If in any word a sequence *aaa* appears, a new word can be formed by replacing the *aaa* by the letter *b.*

C. If in any word the sequence *bbb* appears, a new word can be formed by omitting *bbb.*

D. Given any word, a new word can be formed by writing down the sequence that constitutes the given word twice.

For example, by (D), *aa* is a word, and by (D) again, *aaaa* is a word. Hence by (B) *ba* is a word, and by (A) *bab* is also a word. Again, by (A), *babb* is a word, and so by (D), *babbbabb* is also a word. Finally, by (C) we find that *baabb* is a word.

Prove that in this language *baabaabaa* is not a word.

Comment: Our Problem Editor, Professor George Berzsenyi proposed this problem. It is similar to a problem mentioned in Mathematical Challenges II, published by the Scottish Mathematical Council in 1995.

Solution 1 for 1/4/13 by Jenna Le (12/MN):

(B) and (D) are the only rules that affect the number of times the letter *a* appears in your word.

Suppose you have a word in which the latter *a* appears n times. If you apply rule (B), you will get a new word in which the letter *a* appears $n-3$ times. If you apply rule (D) instead, you will get a new word in which the letter *a* appears $2n$ times.

Unless n is a multiple of three, neither $n-3$ nor $2n$ can be a multiple of 3, by Lemmas 1 and 2 below. Therefore, unless you start out with a word in which the number of times the letter *a* appears is a multiple of three, you will never be able to form a word in which the letter *a* appears a multiple of three times. You start out with the word *a*, which contains one letter *a*, and one is not a multiple of three. Therefore, you will never be able to form the word *baabaabaa*, since *baabaabaa* contains the letter *a* six times, and six is a multiple of three.

QED

Lemma 1. *If 3 divides* $n-3$*, then 3 divides* n.

Suppose $3|(n-3)$. Then there exists an integer x such that $n-3 = 3x$. If follows that $n = 3x+3 = 3(x+1)$, so $3|n$.

Lemma 2. *If 3 divides* $2n$*, then 3 divides* n.

Suppose $3|(2n)$. Then the Fundamental Theorem of Arithmetic implies that, since 3 does not divide 2 and 3 is prime, $3|n$.

Solution 2 for 1/4/13 by Mauro Braunstein (12/FL)

We have here an axiomatic system with axiom *a* and rules of inference A, B, C, and D:

AXIOM: *a* is a word.

RULES: A. If *x* is a word, so is *xb*.

B. If *xaaay* is a word, so is *xby*.

C. If *xbbby* is a word, so is *xy*.

D. If *x* is a word, so is *xx*.

In the above rules, *x* and *y* stand for strings of letters, which could be the empty string $\varnothing$. Notice that $\varnothing$ is not postulated as a word. We will show you that it isn't a word at all, and even its addition to the set of words would still keep *baabaabaa* from being a word.

We will focus on the number of *a*'s for our problem, so let $a(x)$ be the number of times *a* occurs in *x*. For example, $a(baabaabaa) = 6$. Rule A adds a *b* to a word, so the number of *a*'s clearly remains unchanged. Rule B replaces three *a*'s by a *b*, so the number of *a*'s decreases by three. Rule C simply removes three *b*'s, so it also leaves the number of *a*'s unchanged. Rule D duplicates the word, therefore clearly doubling the number of *a*'s. If we let $n = a(x)$,

$$a(A(x)) = n$$
$$a(B(x)) = n-3$$
$$a(C(x)) = n$$
$$a(D(x)) = 2n$$

$a(a) = 1$, and from there we want to create a case where $a(x) = 6$. What order of multiplying by two and subtracting three will take 1 to 6? None. Consider n (mod 3). Subtracting 3 in rule B will leave n (mod 3) unchanged. Look at doubling. We are given 1; $2 \cdot 1 \equiv 2$ (mod 3), and $2 \cdot 2 \equiv 1$ (mod 3), so doubling will yield a repetitive cycle: 1, 2, 1, 2, …. However, $6 \equiv 0$ (mod 3), and no amount of doubling will yield 0 from 1. Since $a(baabaabaa) = 6$, it is impossible to construct it from a string m such that $a(m) = 1$, particularly $m = a$. Likewise, $a(\varnothing) = 0$, so $\varnothing$ cannot be constructed either.

If $\varnothing$ were accepted as an axiom, no rule would allow an increase in its *a*'s. A and C keep the number of *a*'s fixed at 0, D doubles the number of *a*'s from 0 to 0, and B cannot be used because it needs at least three *a*'s, an impossible requirement since we started with zero *a*'s. Therefore, the word *baabaabaa* would not be possible even under this new system.

2/4/13. Let $f(x) = x \cdot \lfloor x \cdot \lfloor x \cdot \lfloor x \rfloor \rfloor \rfloor$ for all positive real numbers *x*, where $\lfloor y \rfloor$ denotes the greatest integer less than or equal to *y*.

(1) Determine *x* so that $f(x) = 2001$.

(2) Prove that $f(x) = 2002$ has no solution.

Comment: This problem was suggested by Professor Bela Bajnok of Cornell University. It is adapted from a Chech-Slovakian problem given a few years ago.

Solution 1 to 2/4/13 by David Barmore (10/IL)

First, we will determine x such that $f(x) = 2001$. For $x \geq 1$, $f(x)$ is strictly increasing. Since $f(6.9965) = 2000.999 < 2001$ and $f(6.9966) = 2001.0276 > 2001$, if $f(x) = 2001$ then $6.9965 < x < 6.9966$. Also, since $x = \dfrac{f(x)}{\lfloor x \lfloor x \lfloor x \rfloor \rfloor \rfloor}$ by definition of $f(x)$, x is rational for all integral values of $f(x)$. For our problem, $x = \dfrac{2001}{\lfloor x \lfloor x \lfloor x \rfloor \rfloor \rfloor}$. For $6.9965 < x < 6.9966$, $\lfloor x \lfloor x \lfloor x \rfloor \rfloor \rfloor = 286$, so x could only be 2001/286, which checks.

Now, we will prove that no x exists such that $f(x) = 2002$. First, if there were some number x such that $f(x) = 2002$, then $x = \dfrac{2002}{\lfloor x \lfloor x \lfloor x \rfloor \rfloor \rfloor}$. Clearly, since $f(\frac{2001}{286}) = 2001 < 2002$ and $f(7) = 2401 > 2002$ and $f(x)$ is strictly increasing on this interval, $\frac{2001}{286} < x < 7$. For x on this interval, $\lfloor x \rfloor = 6$, $\lfloor x \lfloor x \rfloor \rfloor = \lfloor 6x \rfloor = 41$ since $\frac{2001}{286} > \frac{41}{6}$, $\lfloor x \lfloor x \lfloor x \rfloor \rfloor \rfloor = \lfloor 41x \rfloor = 286$ since $\frac{2001}{286} > \frac{286}{41}$. So x could only be 2002/286 = 7. However, $f(7) \neq 2002$. Since we have arrived at a contradiction, our supposition that there exists x such that $f(x) = 2002$ must be false. Therefore, there does not exist x such that $f(x) = 2002$.

Solution 2 to 2/4/13 by Agustya Mehta (10/OH)

Part 1.

It is easy to see that $f(x)$ increases as x increases, because $\lfloor x \lfloor x \lfloor x \rfloor \rfloor \rfloor$ either remains the same or increases when x increases.

We see that $f(6) = 6^4 = 1296$ and $f(7) = 7^4 = 2401$, so if $f(x) = x \lfloor x \lfloor x \lfloor x \rfloor \rfloor \rfloor =$ 2001, then $6 < x < 7$.

Let $m = \lfloor x \lfloor x \lfloor x \rfloor \rfloor \rfloor$. m is an integer by the definition of $\lfloor\ \rfloor$. So, $f(x) = x \cdot m = 2001$ or $x = 2001/m$. Since $x < 7$, we have $286 \leq m$ (Inequality 1). Also since $6 < x < 7$,

$$\lfloor x \rfloor = 6$$

$$x \cdot \lfloor x \rfloor < 42$$

$$\lfloor x \cdot \lfloor x \rfloor \rfloor \leq 41 \qquad \text{(Inequality 2)}$$

$$x \cdot \lfloor x \cdot \lfloor x \rfloor \rfloor < 287$$

$$\lfloor x \cdot \lfloor x \cdot \lfloor x \rfloor \rfloor \rfloor \leq 286$$

So by both the inequalities, $286 \le m \le 286$; thus, $m = 286$. Thus $x = 2001/286$.

Part 2.

As we have shown in part 1, $f(6) = 6^4 = 1296$, $f(7) = 7^4 = 2401$, and 1296 < 2002 < 2401. So if $f(x) = 2002$ has a solution x, it must satisfy $6 < x < 7$. This is the same condition that gave us Inequality 2 in part 1: $\lfloor x \cdot \lfloor x \cdot \lfloor x \rfloor \rfloor \rfloor \le 286$. So $x \cdot \lfloor x \cdot \lfloor x \cdot \lfloor x \rfloor \rfloor \rfloor < 2002$. Thus, $f(x) = 2002$ has no solution.

Solution 3 by Daniel McLaury (11/OK)

x must be a rational number, since we are proposing to multiply it by an integer and get an integer result, so let $x = n + \frac{a}{b}$, where n, a, and b are integers and $0 < a < b$. Then we know that $n = 6$, since if we plug in an x-value less than 6, we'll get a number less than $6^4 = 1296$, and if x is 7 or more, we get $7^4 = 2401$ or more.

Start from inside the nested floor brackets. Obviously, $\lfloor x \rfloor = \left\lfloor 6 + \frac{a}{b} \right\rfloor = 6$. At the next floor bracket, we get $\left\lfloor 36 + (6)(\frac{a}{b}) \right\rfloor$. Since $0 < \frac{a}{b} < 1$, we know that $0 < (6)(\frac{a}{b}) < 6$, so that it becomes an integer between 0 and 5.

Suppose $\left\lfloor (6)(\frac{a}{b}) \right\rfloor$ is 4, which means $\frac{a}{b} < \frac{5}{6}$. We would get 40 for $\left\lfloor 36 + (6)(\frac{a}{b}) \right\rfloor$ and then multiplying by x would give $240 + (40)(\frac{a}{b})$. The floor of that has a maximum value of 273, since $\frac{a}{b} < \frac{5}{6}$. Now 2001 is more than 7 times 273, so obviously this is too small. Therefore, by contradiction, we know that $\left\lfloor (6)(\frac{a}{b}) \right\rfloor = 5$.

So we have that $\lfloor x \lfloor x \rfloor \rfloor = 41$, which means that $\lfloor x \lfloor x \lfloor x \rfloor \rfloor \rfloor = 246 + (41)(\frac{a}{b})$. Now 2001 divided by 7 is just less than 286, so anything smaller than that won't work. Since $\frac{a}{b} < 1$, $(41)(\frac{a}{b}) < 41$, so $\left\lfloor (41)(\frac{a}{b}) \right\rfloor \le 40$. But we need 286, so it must be at least 40. So it is 40.

So finally, $x \lfloor x \lfloor x \lfloor x \rfloor \rfloor \rfloor$ is simply $(6 + \frac{a}{b})(286) = 1716 + (286)(\frac{a}{b})$. We can set $b = 286$, and reduce $\frac{a}{b}$ later if necessary. We get $1716 + a = 2001$, giving $a = 285$. So our final answer is $6 + \frac{285}{286}$.

Letting x be anything between $6 + \frac{285}{286}$ and 7 would still give $f(x) = 286x$, which could not be an integer result. And $f(7) = 7^4$ is much larger than 2002, so it is impossible to get 2002.

I don't like this problem for some reason. There just seems to be way too much brute force arithmetic. Any problem with an answer like $6 + \frac{285}{286}$ just seems to be severely contrived and lacks any real beauty.

Comment by Erin Schram on Mr. McLaury's last observation: We select our problems for solutions of appropriate difficulty that require mathematical understanding. We would love to have beauty in every problem, but given the limited pool of problems that we can beg, borrow, or invent, sometimes we have to make do without beauty. However, our problems are not contrived in the sense of deliberately selecting ugly answers to add more difficulty. We want the mathematics of the problems to be as bold and clear as possible.

3/4/13. Let f be a function defined on the set of all integers, and assume that it satisfies the following properties:

A. $f(0) \neq 0$;

B. $f(1) = 3$; and

C. $f(x)f(y) = f(x+y) + f(x-y)$ for all integers x and y.

Determine $f(7)$.

Comment: This problem is based on a similar problem used in 1987 for the selection of Hungary's IMO team, and was proposed by Prof. Berzsenyi.

Solution 1 for 3/4/13 by George Khachatryan (11/TX)

$f(1) \cdot f(0) = f(1) + f(1) \quad \Rightarrow \quad 3 \cdot f(0) = 3 + 3 \quad \Rightarrow \quad f(0) = 2$

$f(1) \cdot f(1) = f(2) + f(0) \quad \Rightarrow \quad 3 \cdot 3 = f(2) + 2 \quad \Rightarrow \quad f(2) = 7$

$f(2) \cdot f(1) = f(3) + f(1) \quad \Rightarrow \quad 7 \cdot 3 = f(3) + 3 \quad \Rightarrow \quad f(3) = 18$

$f(3) \cdot f(1) = f(4) + f(2) \quad \Rightarrow \quad 18 \cdot 3 = f(4) + 7 \quad \Rightarrow \quad f(4) = 47$

$f(4) \cdot f(3) = f(7) + f(1) \quad \Rightarrow \quad 47 \cdot 18 = f(7) + 3 \quad \Rightarrow \quad f(7) = 843$

so $f(7) = 843$.

However, what is interesting about $f(n)$ is that it has a closed form. The sequence we saw actually is special. Over the positive integers, $f(n-1) \cdot f(1) = f(n) + f(n-2)$, which implies $f(n-1) \cdot 3 = f(n) + f(n-2)$ and

$$f(n) = 3f(n-1) - f(n-2).$$

This is the recursive definition for the even bisection of the Lucas numbers, $L(n)$. The Lucas number $L(n)$ is defined as $L(0) = 2$, $L(1) = 1$, and $L(n) = L(n-1) + L(n-2)$. The sequence is: 2, 1, 3, 4, 7, 11,18, 29, 47, 76, 123, …. (c.f. the Fibonacci numbers).

The even bisection of the Lucas numbers, which are our $f(n)$, has a closed form: $f(n) = \left(\frac{1+\sqrt{5}}{2}\right)^{2n} + \left(\frac{1-\sqrt{5}}{2}\right)^{2n}$. Setting $n = 7$ indeed yields 843.

4/4/13. A certain company has a faulty telephone system that sometimes transposes a pair of adjacent digits when someone dials a three-digit extension. Hence a call to $x318$ would ring at either $x318$, $x138$, or $x381$, while a call received at $x044$ would be intended for either $x404$ or $x044$. Rather than replace the system, the company is adding a computer to deduce which dialed extensions are in error and revert those numbers to their correct form. They have to leave out several possible extensions for this to work. What is the greatest number of three-digit extensions the company can assign under this plan?

Comment: This problem was proposed by Dr. Peter Anspach of the National Security Agency.

Solution 1 for 4/4/13 by Tamara Broderick (11/OH)

How many three-digit extensions can be assigned so that the computer can deduce the correct extension every time?

For extensions of the form *aaa*:

All possible outputs of the faulty system: *aaa*

Therefore, every extension of the form *aaa* (10 total) may continue to be used as it will not be confused with any other extension.

For extensions of the form *aab*:

Possible extensions of form *aab*	Possible outputs of the system
aab	*aab, aba*
aba	*aba, baa, aab*
baa	*baa, aba*

Since there are no two extensions that have no shared possible outputs, all of these extensions could be confused with the others, and only one can, thus, be used by the computer system. Therefore, 1/3 of all extensions of this form can be assigned.

For extensions of the form *abc*:

Possible extensions of form *abc*	Possible outputs of the system
abc	*abc, bac, acb*
acb	*acb, cab, abc*
bac	*bac, abc, bca*
bca	*bca, cba, bac*
cab	*cab, acb, cba*
cba	*cba, bca, cab*

Each extension can be confused with two other extensions. Therefore, the maximum number of extensions that may be verified as unique by the new computer out of 6 such extensions is 2. For example, *abc* and *cba* share no possible outputs in common. Thus, 2/6 or 1/3 of these extensions can be used by the computer.

Since 1/3 of all forms besides form *aaa* are usable, 1/3 of their remaining 990 total three-digit extensions (after subtracting all 10 of form *aaa*) are usable. Therefore, the greatest number of assignable extensions is $10 + 990/3 = 340$.

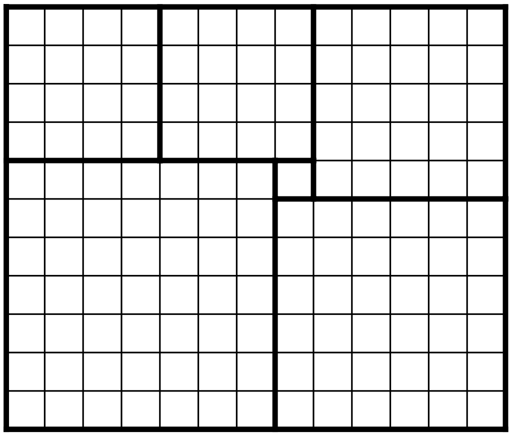
Solution 2 for 4/4/13 by David Galkowski (10/NY)

1. The possible extension numbers can be divided into 3 groups: extension numbers with three different digits, extension numbers with two different digits with one repeated, and extension numbers with one digit used three times.
2. Extension numbers with three different digits are the largest group. There are 10 choose 3 ways to pick three different digits. Each set of three digits has two possible extensions that will always produce different numbers when garbled. For example, if the digits 3, 2, and 1 were chosen, we could chose x123 and x321. When x123 was dialed, the computer would receive 123, 213, or 132. When x321 was dialed, the computer would receive 321, 312, or 231. Since neither share numbers, yet all possible combinations of 3, 2, and 1 are used, there are two possible extensions for each extension number with three different digits. the number of extension numbers with three different digits is $2 \cdot \binom{10}{3} = 240$.
3. The number of extension numbers with two different digits, one repeated once, is $10 \cdot 9$. There are 10 choices for the number that repeats and 9 choices for a different number. If two 3's and a 1 were chosen for an extension, only one number could be used, since all three possible extensions could share a possible outcome of 313.
   x133 could change to 313 or 133.
   x313 could change to 133, 313, or 331.
   x331 could change to 313 or 331.
   The number of extension numbers with two different digits is $10 \cdot 9 = 90$.
4. The number of extension numbers with 1 digit repeated twice more is 10 since there are only 10 digits and there is only one possible extension for each digit.
5. The greatest number of three-digit extensions the company can assign is $240 + 90 + 10 = 340$ extensions.

5/4/13. Determine the smallest number of squares into which one can dissect a 11×13 rectangle, and exhibit such a dissection. The squares need not be of different sizes, their bases should be integers, and they should not overlap.

Comment: We thank Prof. Berzsenyi for proposing this interesting problem. For an excellent introduction to the topic of dissecting rectangles into squares, the reader is referred to Chapter 17 of *Scientific American's Second Book of Mathematical Puzzles and Diversions* by Martin Gardner.
Comment by Erin Schram: To a mathematician, “determine” means “find and verify”. Proving that six is the smallest number of squares in a dissection is half the work in the problem, and a solution would be incomplete without such a verification.

Solution 1 for 5/4/13 by Paul H. Ryu (10/CA)



This figures shows a dissection with six squares. Of the six squares, two are of size 4×4, and the other four squares are of sizes 1×1, 5×5, 6×6, and 7×7. Since the dissection is valid, we only have to verify that there are no possible dissections with fewer than 6 squares.

To do that, we have to first figure out the set of whole numbers whose squares sum up to the area of the rectangle, which is 143.

With the aid of a computer program, one can find sets of whole numbers whose squares sum up to the area of the rectangle. There are no sets of two or three whole numbers whose squares sum to 143. There are five sets of four whole numbers whose squares sum to 143. They are {1, 5, 6, 9}, {2, 3, 3, 11}, {2, 3, 7, 9}, {3, 3, 5, 10}, and {3, 6, 7, 7}. However, none of them actually work, because the squares do not fit together. That is because the combined length of the two largest squares exceeds 13; thus, two of them cannot be laid on the 11×13 rectangle without exceeding the boundary. The two squares have to be laid down somehow, but if two of them have sides summing greater than 13, their sides opposite each other will be more than 13 apart.

Now we find sets of five whole numbers whose squares sum to 143. There are eight of them: {1, 1, 2, 4, 11}, {1, 1, 4, 5, 10}, {1, 2, 5, 7, 8}, {1,3, 4, 6, 9}, {2, 4, 5, 7, 7}, {2, 5, 5, 5, 8}, {3, 3, 3, 4, 10}, and {3, 3, 5, 6, 8}. None can actually be arranged into an 11×13 rectangle. The squares simply do not fit. As before, the sum of the largest two squares exceeds 13, except with the {2, 5, 5, 5, 8} case, in which the 5×5 squares cannot fill the gaps left by the 8×8 square.

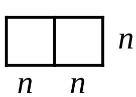
Thus, we see that our dissection, which uses the set {1, 4, 4, 5, 6, 7}, is optimal, and the smallest number of squares into which one can dissect an 11×13 rectangle is six squares.

Solution 2 for 5/4/13 by Xuan Wu (12/CA)

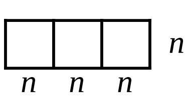
Testing for smallest number of squares:

1? No. Biggest square that will fit in 11×13 is 11×11, which does not cover the remaining 2×11.

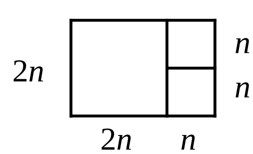
2? No. Only configuration of two squares that make a rectangle is the figure to the right. The length has to be 2 times longer than the width, $L = 2W$. However, 13 is not $2 \cdot 11$, so an 11×13 rectangle cannot be formed by 2 squares.



3? No. Two configurations:

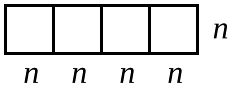


$L = 3W$

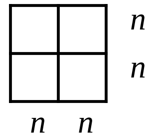


$2L = 3W$

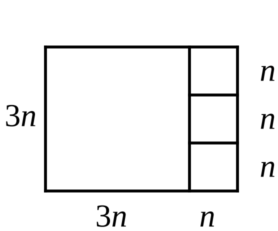
4? No. Five configurations:



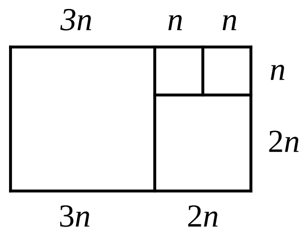
$L = 4W$



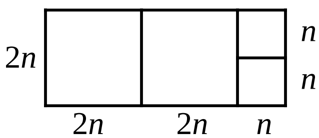
$L = W$



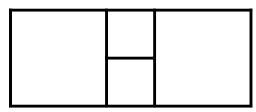
$3L = 4W$



$3L = 5W$

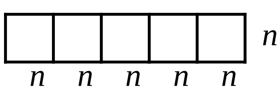


$2L = 5W$

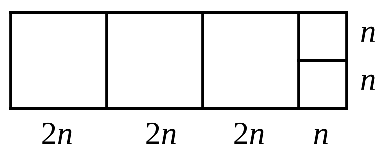


Considered to be the same as the case to the left; anytime the same squares can be rearranged to form the same dimensions, it will be considered the same configuration.

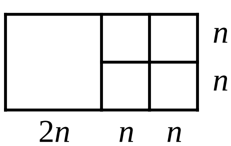
5? No. Eleven configurations:



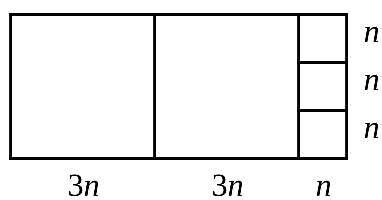
$L = 5W$



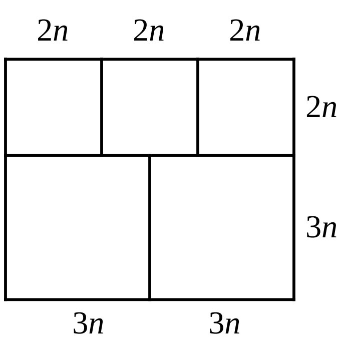
$2L = 7W$



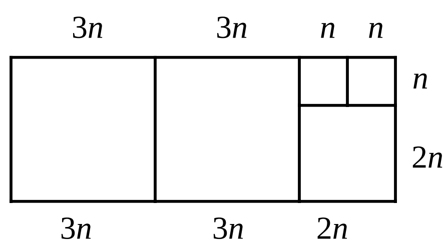
$L = 2W$



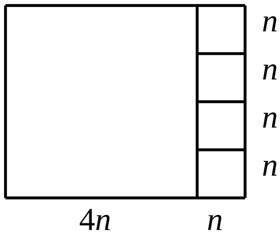
$3L = 7W$



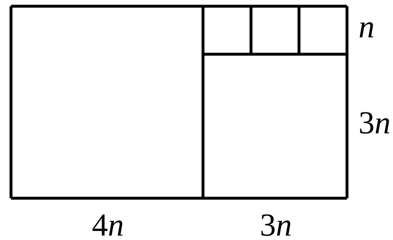
$5L = 6W$



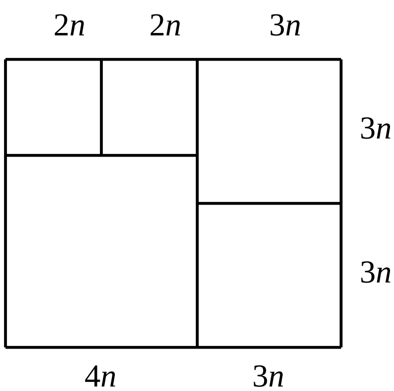
$3L = 8W$



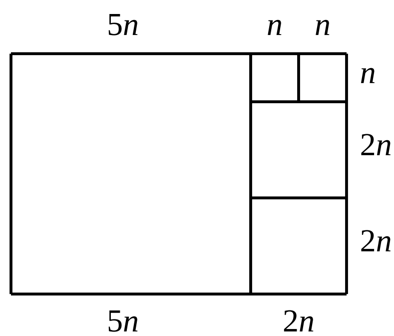
$4L = 5W$



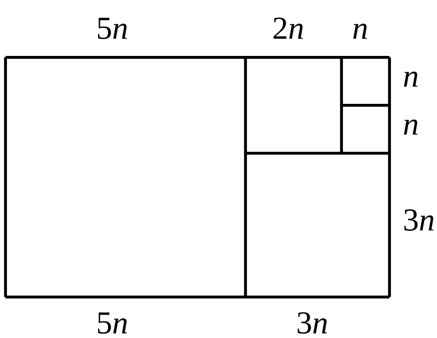
$4L = 7W$



$6L = 7W$

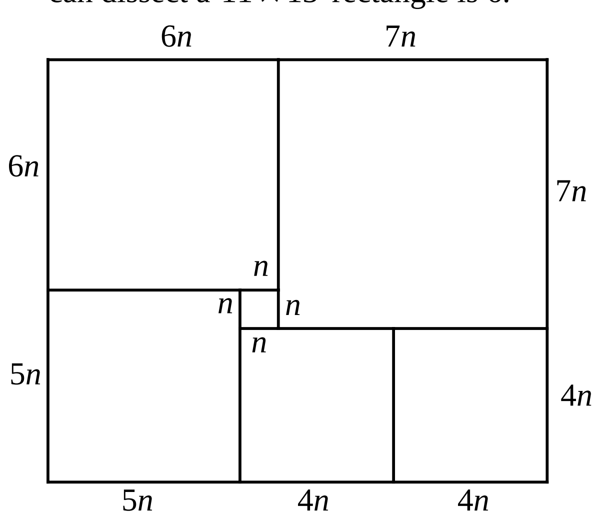


$5L = 7W$



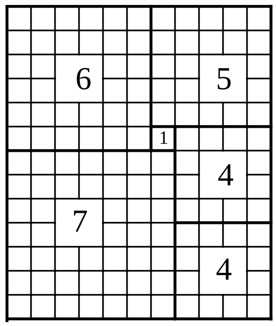
$5L = 8W$

6? Yes. One configuration, shown below, works. The smallest number of squares into which one can dissect a 11×13 rectangle is 6.



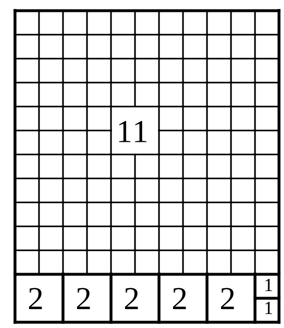
$11L = 13W$

Solution 3 for 5/4/13 by Robert Cordwell (9/NM)



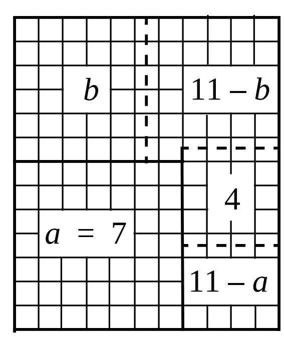
This is the solution with six squares.

It is relatively easy to prove that at least six squares are needed. We know that there must be at least one square at each corner for a minimum solution, since we know that a solution with 6 squares exists and that a solution with one square of size 11 covering two corners takes 8 squares. We prove that there must be 6 squares by showing that a solution with five or fewer squares is not possible.

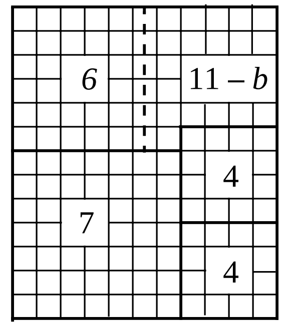


Notice that all the sides of the rectangles must be covered completely by the sides of the squares, and each corner square of length x covers x units on one of the rectangle's sides of length 13 and x units on one of the rectangle's sides of length 11. It should be fairly clear that with five or fewer squares at least three sides of the rectangle must be covered solely by the corner squares, including at least one 13-long side. Let those two squares on the long side have sides of lengths a and b. In order to have at most five squares, the other two corner squares must have lengths of $11-a$ and $11-b$, which sum up to 9, leaving a gap of 4 on the other side. Thus, there must be a square of side length 4 (it could be done with more squares, but with fives squares we can use only one non-corner square) on a 13-long side.

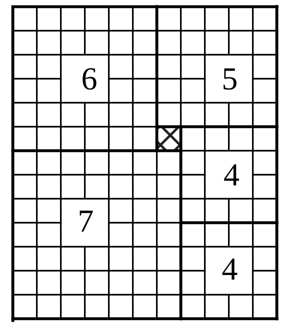
We must now try to place the four corner squares so that they cover the rest of the rectangle. Notice that a square opposite the square of length 4 must have side length 7 to avoid placing a sixth square between them.



This in turn means that we have squares on the two adjacent corners of side lengths 4 and 6.



Finally, this means that there must be a square with side length 5 in the last corner.



However, when one adds up the areas of all five large squares: $16 + 16 + 25 + 36 + 49$, one finds that it is 142, one short of the area of the rectangle, $11 \times 13 = 143$. Thus, there is no solution with five squares. By adding the one missing square, we have instead found our solution with six squares.