USA Mathematical Talent Search

PROBLEMS / SOLUTIONS / COMMENTS
Round 3 - Year 13 - Academic Year 2001-2002

Erin Schram, Grader

1/3/13. We will say that a rearrangement of the letters of a word *has no fixed letters* if, when the rearrangement is placed directly below the word, no column has the same letter repeated. For instance, the blocks of letters below show that $E\ S\ A\ R\ E\ T$ is a rearrangement with no fixed letters of $T\ E\ R\ E\ S\ A$, but $R\ E\ A\ S\ T\ E$ is not.

$$\begin{array}{cccccc} T & E & R & E & S & A \\ E & S & A & R & E & T \end{array} \qquad\qquad \begin{array}{cccccc} T & E & R & E & S & A \\ R & E & A & S & T & E \end{array}$$

How many distinguishable rearrangements with no fixed letters does $T\ E\ R\ E\ S\ A$ have? (The two E's are considered identical.)

Comment: Our Problem Editor, Professor George Berzsenyi, suggested this problem.

Solution 1 for 1/3/13 by Dmitriy Yegoshin (12/CA):

First let's position our E's since they will present the biggest problem. There are only 6 valid arrangements of the 2 E's where both of them are in different positions than in *TERESA*. They are:

1 $E * E * * *$
2 $E * * * E *$
3 $E * * * * E$
4 $* * E * E *$
5 $* * E * * E$
6 $* * * * E E$

Now that we have the E's positioned, we can go ahead and position the other 4 letters. But since the other 4 letters are not repeated in *TERESA*, the number of different placements for those 4 letters is the same for each of the 6 combinations of E's.

Let's count all the possible combinations for one of the E's combinations:

T E R E S A	T E R E S A	T E R E S A
E T E R A S	E R E A T S	E A E T R S
E T E S A R	E S E T A R	E A E R T S
E T E A R S	E S E R A T	E A E S T R
E R E T A S	E S E A T R	E A E S R T
E R E S A T	E S E A R T	

We get 14 different combinations.

So now since each of the 6 E's combinations has 14 sub-combinations, we multiply $6 \times 14 = 84$.

Therefore, there are a total of 84 distinguishable arrangements.

Solution 2 for 1/3/13 by Elena Ruse (10/NY):

To get a suitable rearrangement, we must move each of the E's to one of the positions 1, 3, 5, or 6. There are $C_4^2 = 6$ ways to do that. Say we move the two E's to where the letters X, Y, which are two of the letters in $\{T, R, S, A\}$, were originally. Our problem is now to place the other two letters, which we'll call Z and W, in 2 of the remaining 4 positions so that neither is in its original position. (the E's are now where X and Y were originally, so, of course, neither X nor Y will remain fixed no matter where we put them). Now, we can either move Z to where W was, in which case we can move W to any of the 3 remaining positions, or move Z to the original positions of one of the E's (which we can do 2 ways), in which case we must move W either to where Z was or where the other E was. So, once we've placed the E's (which we can do in 6 ways), there are $3 + 2 \cdot 2 = 7$ ways to place Z and W, and then 2 ways to place X and Y in the remaining 2 positions. The total number of suitable rearrangements is, therefore, $6 \cdot 7 \cdot 2 = 84$.

Solution 3 for 1/3/13 by Dheera Venkatraman (12/NJ):

There are $6! = 720$ ways of arranging six letters. Since two are identical (there are two E's), there are 360 distinct arrangements. Of these:

- One-sixth (360/6 = 60) will have T as the first letter and, therefore, have a fixed letter. These are immediately excluded, leaving 300 arrangements.
- One-sixth of the original 360 arrangements will have R as the third letter. Of this one-sixth, one-fifth will also have T as the first letter, which has already been counted. This creates $60 - 60/5 = 48$ rearrangements which can be eliminated. This leaves $300 - 48 = 252$ arrangements.
- One-sixth of the original 360 will have S as the fifth letter. Following similar logic: one-fifth of these will have R as the third letter, which has been counted; another one-fifth will have T as the first letter; but one-fourth of each of those will have R as the third letter and T as the first letter, and this intersection of the two sets (the set with R as the third letter and the set with T as the first letter) must not be subtracted twice. This creates $60 - (2 \times (60/5) - (60/5)/4) =$ 39 arrangements which can be eliminated, leaving a total of 213 arrangements.
- Following this pattern for those with the sixth letter being A, $60 - (3 \times (60/5) - (3 \times ((60/5)/4) - ((60/5)/4)/3)) = 32$ arrangements which are eliminated, leaving 181 arrangements.
- To eliminate the occurrences of the E's, a slightly different pattern must be used. Here, because there are two E's, one-third ($360/3 = 120$) of the original sequence bear E as the second letter. Of those, one-fifth will have A as the sixth letter, one-fifth will have S as the fifth letter, one-fifth will have R as the third letter, and one-fifth will have T as the first letter. Also, of the one-fifth that have A as the sixth letter, one-fourth of those will also have S as the fifth letter, and so on. This gives $120 - (4 \times (120/5) - (6 \times ((120/5)/4) - (4 \times (((120/5)/4)/3) - (((120/5)/4)/3)/2)))$ = 53 arrangements with E as the second letter that have not been eliminated yet. Likewise, there are 53 arrangements with E as the fourth letter that have not been eliminated yet. This makes $53 + 53 = 106$; but that counts arrangements with E's as the second letter and fourth letter twice. Of the original 360, one-fifth of one-third, 24 arrangements, have E's in both places. Of those, one-fourth will have A as the sixth letter (since two letters are in place

already, there are only four places of the *A*), one-fourth will have *S* as the fifth letter, and so on. This gives

$$24 - (4 \times (24/4) - (6 \times ((24/4)/3) - (4 \times (((24/4)/3)/2) - (((24/4)/3)/2)/1))) = 9$$

arrangements not to subtract twice. $53 + 53 - 9 = 97$, leaving 84 arrangements.

There are, therefore, 84 permutations of the letters *TERESA* that contain no fixed letters.

An interesting note: Note the coefficients of each term of the expressions of the number of permutations eliminated in each step of the sequence for letters of a single occurrence:

$[1] \times 60$ eliminations for *T*

$[1] \times 60 - [1] \times (60/5)$ for *R*

$[1] \times 60 - ([2] \times (60/5) - [1] \times (60/5/4))$ for *S*

$[1] \times 60 - ([3] \times (60/5) - ([3] \times (60/5/4) - [1] \times (60/5/4/3)))$ for *A*

Note the coefficients which follow Pascal's triangle. To illustrate why this occurs, not the derivation of the coefficients of the last step shown here for *A*: One must first count the occurrences of A itself in its fixed position; then the number of times each of the other letters appears fixed by itself (*i.e., T, R,* or *S*), followed by the number of times each group of two letters appears fixed (*i.e.,* after subtracting the appearances of *T, R,* and *S,* you need to make sure you do not subtract the occurrences of *T* and *R*, *R* and *S*, or *S* and *T* twice), followed by each group of three letters, and finally the single group of all four letters.

2/3/13. Without computer assistance, find five different sets of three positive integers $\{k, m, n\}$ such that $k < m < n$ and $\dfrac{1}{k} + \dfrac{1}{m} + \dfrac{1}{n} = \dfrac{19}{84}$.

[To think about, but not a part of the problem: How many solutions are there?]

Comment: Our Problem Editor, Professor George Berzsenyi, suggested this problem too. There are 22 different sets of {*k, m, n*}. The second solution below lists them all.

Solution 1 for 2/3/13 by Boris Alexeev (10/GA):

Answer: (5, 60, 105), (6, 21, 84), (6, 28, 42), (7, 14, 84), (7, 21, 28)

We will attempt to find only five sets, not all of them. Our first step will be to look for solutions of the form {*k, m, n*} where *k, m,* and *n* are divisors of 84, the denominator of the righthand side. So we let $a = 84/k$, $b = 84/m$, and $c = 84/n$, and multiply both sides of the equation $1/k + 1/m + 1/n = 19/84$ by 84 to get $a + b + c = 19$ and $a > b > c$. Since *k* is a factor of 84, so is *a*. Likewise for *b* and *c*. $\{a, b, c\} \subset \{1, 2, 3, 4, 6, 7, 12, 14, 21, 28, 42, 84\}$.

It's not hard to see that *a* is either 7, 12, or 14. If *a* is 7, we must find $b > c$ with $b + c = 12$, but this is impossible. If *a* is 12, we must find $b > c$ with $b + c = 7$. There are two such pairs: $b = 6$, $c = 1$ and $b = 4$, $c = 3$. These yield $k = 7$, $m = 14$, $n = 84$ and $k = 7$, $m = 21$, $n = 28$, respectively. If *a* is 14, we must find $b > c$ with $b + c = 5$. There are two such pairs, $b = 4$, $c = 1$ and $b = 3$, $c = 2$. These yield $k = 6$, $m = 21$, $n = 84$ and $k = 6$, $m = 28$, $n = 42$, respectively.

Now, that is only four sets, to find a fifth, we notice that we already have solutions with $k = 6$ and $k = 7$, so we arbitrarily try $k = 5$. We then get the equation $1/m + 1/n = 11/420$. We use the same trick as before, multiplying by 420 this time, to get $d + e = 11$, where d and e are factors of 420. It doesn't take much work at all to find one possible solution: $d = 7$, $e = 4$, yielding a fifth solution of $k = 5$, $m = 60$, $n = 105$.

Note that there are, in fact, many more than five solutions: there are 22.

Solution 2 for 2/3/13 by George Lin (10/NJ):

Since $0 < k < m < n$, $\frac{1}{n} < \frac{1}{m} < \frac{1}{k}$, and therefore,

$$\frac{1}{k} < \frac{19}{84} = \frac{1}{k} + \frac{1}{m} + \frac{1}{n} < \frac{3}{k}$$

From this we get,

$$\frac{84}{19} < k < \frac{3 \cdot 84}{19}.$$

Since k is an integer, this simplifies to $5 \le k \le 13$.

Notice that $\frac{19}{84} = \frac{1}{7} + \frac{1}{12}$. Let us consider $k = 7$. Here, we need only solve m and n such that

$$\frac{1}{m} + \frac{1}{n} = \frac{1}{12}. \qquad (1)$$

From

$$\frac{1}{m} < \frac{1}{12} = \frac{1}{m} + \frac{1}{n} < \frac{2}{m}$$

we have $12 < m < 24$. Since m is an integer, this implies that $13 \le m \le 23$. Solving equation (1) for n, we get

$$n = \frac{12m}{m - 12}.$$

Now, let us find n for $m = 13$ to 23, and see which values produce integers.

m	13	14	15	16	17	18	18	20	21	22	23
n	156	84	69	48	$\frac{204}{5}$	36	$\frac{228}{7}$	30	28	$\frac{132}{5}$	$\frac{276}{11}$
Integer?	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes

Using the same method for other values of k, we can find all the solutions. They are:

k m n	k m n	k m n	k m n
5 39 1820	6 17 1428	7 13 156	8 10 840
5 40 840	6 18 252	7 14 84	8 12 56
5 42 420	6 20 105	7 15 60	
5 45 252	6 21 84	7 16 48	
5 56 120	6 24 56	7 18 36	
5 60 105	6 28 42	7 20 30	
5 70 84		7 21 28	

Therefore, there are 22 integer solutions total.

3/3/13. Suppose $p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$ is a monic polynomial with integer coefficients. [Here *monic* polynomial just means the coefficient of x^n is one.] If $(p(x))^2$ is a polynomial all of whose coefficients are non-negative, is it necessarily true that all the coefficients of $p(x)$ must be non-negative? Justify your answer.

Comment: This problem was created by professor Bruce Reznick of the University of Illinois and was first featured in the 1990 Indiana College Mathematics Competition.

Solution 1 for 3/3/13 by Andrew Newman (11/AL):

No. A counterexample suffices:

$$(x^4 + 2x^3 - x^2 + 3x + 4)^2 = x^8 + 4x^7 + 2x^6 + 2x^5 + 21x^4 + 10x^3 + x^2 + 24x + 16 .$$

The conjecture that non-negative integer coefficients in $(p(x))^2$ implies non-negative coefficients in $p(x)$ is true for all first, second, and third degree $p(x)$, but it is not true in general for any higher degree. We will consider the cases of degrees 1, 2, 3, and 4, and then give a construction for higher degrees.

Case 1: $p(x) = x + a$

$$(p(x))^2 = x^2 + 2ax + a^2$$

$2a \geq 0$ implies $a \geq 0$, so the conjecture is true for first degree $p(x)$.

Case 2: $p(x) = x^2 + ax + b$

$$(p(x))^2 = x^4 + 2ax^3 + (a^2 + 2b)x^2 + 2abx + b^2$$

$2a \geq 0$ implies $a \geq 0$. If $a > 0$, then $2ab \geq 0$ implies $b \geq 0$. If $a = 0$, then $a^2 + 2b \geq 0$ implies $b \geq 0$. So the conjecture is true for second degree $p(x))$.

Case 3: $p(x) = x^3 + ax^2 + bx + c$

$$(p(x))^2 = x^6 + 2ax^5 + (a^2 + 2b)x^4 + (2ab + 2c)x^3 + (b^2 + 2ac)x^2 + 2bcx + c^2$$

$2a \geq 0$ implies $a \geq 0$. Since $2bc \geq 0$, either *b* and *c* are both positive, they are both negative, or at least one is zero. If they are both negative, then $2ab + 2c$ would be negative, a contradiction. If $a > 0$ and *b* or *c* is zero, then $2ab + 2c \geq 0$ makes the other one non-negative. If $a = 0$, we use $a^2 + 2b \geq 0$ and $2ab + 2c \geq 0$ instead. Thus, both *b* and *c* are non-negative, and the conjecture is true for third degree $p(x)$.

Case 4: $p(x) = x^4 + ax^3 + bx^2 + cx + d$

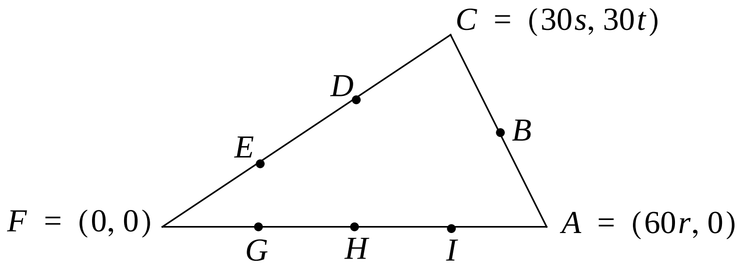
$$(p(x))^2 = x^8 + 2ax^7 + (a^2 + 2b)x^6 + (2ab + 2c)x^5 + (b^2 + 2ac + 2d)x^4$$
$$+ (2ad + 2bc)x^3 + (c^2 + 2bd)x^2 + 2cdx + d^2$$

As before, $a \geq 0$. Since $2cd \geq 0$, if neither of *c* and *d* is zero, they have the same sign. Suppose *c* and *d* are positive and let $b = -1$. Since $a^2 + 2b \geq 0$, we have $a^2 \geq 2$. Let $a = 2$. From $2ab + 2c \geq 0$ we have $c \geq 2$. Let $c = 3$. There are four conditions on *d*. The first is $b^2 + 2ad + 2d \geq 0$, which is satisfied since we assumed *d* is positive. The second is $2ad + 2bc \geq 0$, which is equivalent to $d \geq 1.5$. The third is $c^2 + 2bd \geq 0$, or $d \leq 4.5$. The fourth is $2cd \geq 0$, which we made sure was satisfied when we supposed that *c* and *d* are positive. Thus, any integer value for *d* in [1.5, 4.5], namely 2, 3, or 4, provides a counterexample where all the coefficients of $(p(x))^2$ will be non-negative but $p(x)$ will have a negative coefficient, namely $b = -1$. The counterexample with $d = 4$ appears at the beginning of this solution.

For any degree *n* greater than 4, a counterexample to the conjecture can be obtained by multiplying the above counterexample by x^{n-4}.

It should also be noted that the conjecture is true for a polynomial of any degree that has only real roots, no imaginary or non-real complex roots. Squaring $p(x)$ introduces no new unique roots. Since $(p(x))^2$ has all non-negative coefficients, it follows by Descartes' Rule of Signs that it has no positive roots., and since all its roots are real, it must have only negative and zero roots. $p(x)$ has the same roots (which each root occurring with half the multiplicity) and so may be written as $p(x) = (x + r_1)(x + r_2)\ldots(x + r_n)$ with each r_i being positive or zero, so that when expanded $p(x)$ must have all non-negative coefficients.

4/3/13. As shown in the figure on the right, in ΔACF, B is the midpoint of $\overline{AC}$, D and E divide side $\overline{CF}$ into three equal parts, while G, H, and I divide side $\overline{FA}$ into four equal parts.



Seventeen segments are drawn to connect these six points to one another and to the opposite vertices of the triangle. Determine the points interior to ΔACF at which three or more of these line segments intersect one another.

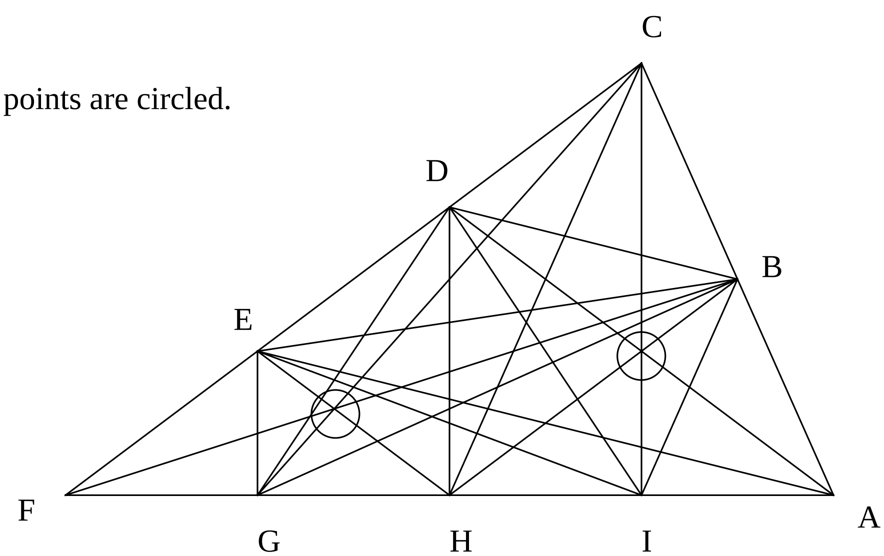
To make grading easier, we have embedded the triangle into the first quadrant with point F at the origin, point C at $(30s, 30t)$, and point A at $(60r, 0)$, where r, s, and t are arbitrary positive real numbers. Please use this notation in your solutions.

Comment: An article about M.C. Escher's work in the April 1998 issue of *Pythagoras*, an excellent journal for high school students in the Netherlands, inspired Professor Berzsenyi to propose this problem. Escher divided the sides of the triangle into 3, 4, and 5 parts and discovered that there are 17 such triple points of intersection within that triangle.

A visual solution by drawing all the segments and looking for triple intersections is a valid proof that there are at most two triple intersections, but the two triple intersections have to be verified, since each seeming triple intersection might instead be three separate intersections very close to each other.

Solution 1 for 4/3/13 by William Groboski (11/WI):

The first thing is to draw the triangle and see where any three points nearly cross.



We can see there are two possible points, where $\overline{GC}$, $\overline{EH}$, and $\overline{FB}$ cross and where $\overline{HB}$, $\overline{IC}$, and $\overline{AD}$ cross.

The next step is to find where the points are:

$$\begin{array}{lll} A = (60r, 0) & D = (20s, 20t) & G = (15r, 0) \\ B = (15s + 30r, 15t) & E = (10s, 10t) & H = (30r, 0) \\ C = (30s, 30t) & F = (0, 0) & I = (45r, 0) \end{array}$$

The next step is to define the lines, this is easy. First you find the slope, and then use the point-slope form of a line. The equations are:

$$\overline{GC} \cong \quad y = \left(\frac{2t}{2s - r}\right)x - \frac{30rt}{2s - r}$$

$$\overline{EH} \cong \quad y = \left(\frac{t}{s - 3r}\right)x - \frac{30rt}{s - 3r}$$

$$\overline{FB} \cong \quad y = \left(\frac{t}{s + 2r}\right)x$$

We will set two right sides equal to each other and solve for x and then input that x to find y. We should do this for each pair to find their crossing.

It is found that they all cross at $(12r + 6s, 6t)$.

We do the same for the other three lines:

$$\overline{HB} \cong \quad y = \left(\frac{t}{s}\right)x - \frac{30rt}{s}$$

$$\overline{IC} \cong \quad y = \left(\frac{2t}{2s - 3r}\right)x - \frac{90rt}{2s - 3r}$$

$$\overline{AD} \cong \quad y = \left(\frac{t}{s - 3r}\right)x - \frac{60rt}{s - 3r}$$

We find where they all intersect and they also all intersect at one point. They all meet at $(30r + 10s, 10t)$.

The points in the interior of the triangle were three or more segments intersect one another are $(12r + 6s, 6t)$ and $(30r + 10s, 10t)$.

Solution 2 for 4/3/13 by Dimitri Pavlichin (11/PA)

There are 17 segments that divide triangle ABC. Of these 17, segments $\overline{EG}$, $\overline{DB}$, and $\overline{IB}$ only cross segments that all intersect at a vertex, thus eliminating them from any possible intersection of at least 3 segments in the interior. Now there are only 14 segments to examine.

A list of possible two-segment intersections was made. Parallel segments, segments that cannot intersect inside the triangle (such as $\overline{DI}$ and $\overline{EH}$) and segments that share a common starting point (such as $\overline{EH}$ and $\overline{EI}$) were not included.

$\overline{AD} \cap \overline{BE}$	$\overline{AE} \cap \overline{BG}$	$\overline{BE} \cap \overline{CG}$	$\overline{BF} \cap \overline{CI}$	$\overline{BG} \cap \overline{DH}$	$\overline{CG} \cap \overline{DI}$
$\overline{AD} \cap \overline{BF}$	$\overline{AE} \cap \overline{BH}$	$\overline{BE} \cap \overline{CH}$	$\overline{BF} \cap \overline{DG}$	$\overline{BG} \cap \overline{DI}$	$\overline{CG} \cap \overline{EH}$
$\overline{AD} \cap \overline{BG}$	$\overline{AE} \cap \overline{CG}$	$\overline{BE} \cap \overline{CI}$	$\overline{BF} \cap \overline{DH}$	$\overline{BG} \cap \overline{EH}$	$\overline{CG} \cap \overline{EI}$
$\overline{AD} \cap \overline{BH}$	$\overline{AE} \cap \overline{CH}$	$\overline{BE} \cap \overline{DG}$	$\overline{BF} \cap \overline{DI}$	$\overline{BG} \cap \overline{EI}$	$\overline{CH} \cap \overline{DI}$
$\overline{AD} \cap \overline{CG}$	$\overline{AE} \cap \overline{CI}$	$\overline{BE} \cap \overline{DH}$	$\overline{BF} \cap \overline{EH}$	$\overline{BH} \cap \overline{CI}$	$\overline{CH} \cap \overline{EI}$
$\overline{AD} \cap \overline{CH}$	$\overline{AE} \cap \overline{DG}$	$\overline{BE} \cap \overline{DI}$	$\overline{BF} \cap \overline{EI}$	$\overline{BH} \cap \overline{DI}$	$\overline{DG} \cap \overline{EH}$
$\overline{AD} \cap \overline{CI}$	$\overline{AE} \cap \overline{DH}$	$\overline{BF} \cap \overline{CG}$	$\overline{BG} \cap \overline{CH}$	$\overline{BH} \cap \overline{EI}$	$\overline{DG} \cap \overline{EI}$
$\overline{AE} \cap \overline{BF}$	$\overline{AE} \cap \overline{DI}$	$\overline{BF} \cap \overline{CH}$	$\overline{BG} \cap \overline{CI}$	$\overline{CG} \cap \overline{DH}$	$\overline{DH} \cap \overline{EI}$

The coordinates of each of the points on the sides of the triangle were derived. For example, the coordinate of point *B* was derived by the midpoint formula:

$X_{mid} = (X_1 + X_2)/2 \qquad Y_{mid} = (Y_1 + Y_2)/2.$

The *x*-values of the coordinates of point *G*, *H*, and *I* were derived by multiplying the *x*-value of point *A*, which was 60*r*, by 1/4, 1/2, and 3/4, respectively, since those points divided segment $\overline{AF}$ into four segments of equal length.

The *x*- and *y*-values of the coordinates of point *D* and *E* were derived by multiplying the *x*- and *y*-values of point *C*, which were $(30s, 30t)$, by 1/3 and 2/3. respectively, since these points divided segment $\overline{CF}$ into segments of equal length, and if perpendicular lines were drawn from points *C*, *D*, and *E* onto the *x*- and *y*-axes, each axis would also have been divided into segments of equal length.

Thus, the coordinates are as listed: $B = (15s + 30r, 15t)$, $D = (20s, 20t)$, $E = (10s, 10t)$, $G = (15r, 0)$, $H = (30r, 0)$, $I = (45r, 0)$.

The equations of the lines that contain these line segments were derived. Only 14 equations were written, since 3 segments were excluded as described above.

$\overline{AD} \cong tx + (3r - s)y = 60rt$	$\overline{CH} \cong tx + (r - s)y = 30rt$
$\overline{AE} \cong tx + (6r - s)y = 60rt$	$\overline{CI} \cong 2tx + (3r - 2s)y = 90rt$
$\overline{BE} \cong tx - (6r + s)y = -60rt$	$\overline{DG} \cong 4tx + (3r - 4s)y = 60rt$
$\overline{BF} \cong tx - (2r + s)y = 0$	$\overline{DH} \cong 2tx - (3r - 2s)y = 60rt$
$\overline{BG} \cong tx - (r + s)y = 30rt$	$\overline{DI} \cong 4tx + (9r - 4s)y = 180rt$
$\overline{BH} \cong tx - sy = 30rt$	$\overline{EH} \cong tx + (3r - s)y = 30rt$
$\overline{CG} \cong 2tx - (r - 2s)y = 30rt$	$\overline{EI} \cong 2tx + (9r - 2s)y = 90rt$

These equations were used in systems of 3 equations to see whether the 3 segments contained in a system of 3 equations intersect in a common point. (*Note from Erin Schram: though Mr. Pavlichin does not mention it, checking his list of 48 line segment intersections, I find 39 cases were 3 line segments each intersect the other two, though not necessarily at a common point. So he had to check 39 systems of 3 equations. It would be about as easy to simply find the 48 points of intersection for all the pairs of line segments and check for matching points.*)

The intersection of $\overline{AD}$, $\overline{BH}$, and $\overline{CI}$ is a common point: $(30r + 10s, 10t)$.

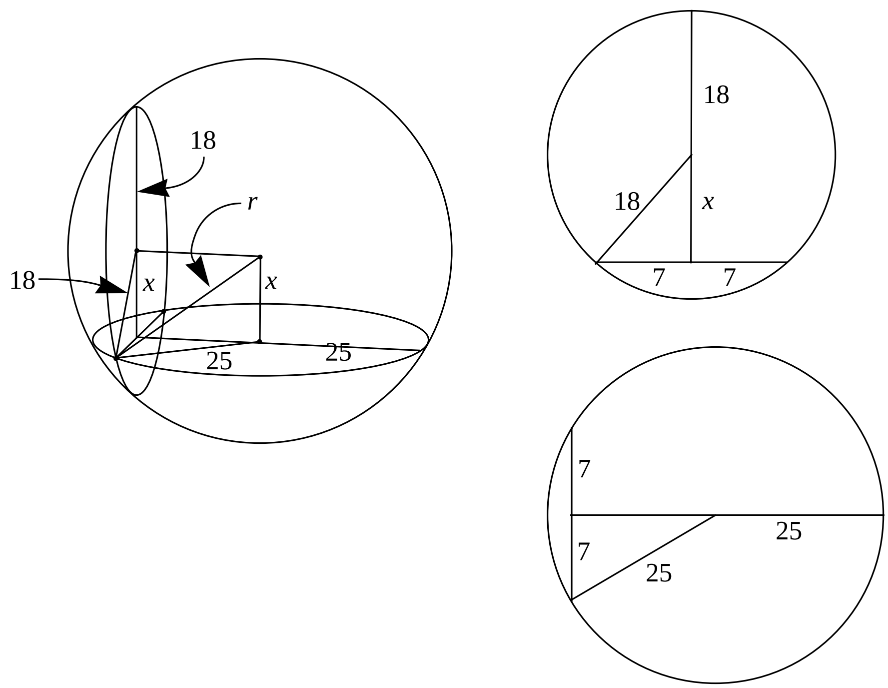
The intersection of $\overline{BF}$, $\overline{CG}$, and $\overline{EH}$ is a common point: $(12r + 6s, 6t)$.
No other points of triple intersection were found.

5/3/13. Two perpendicular planes intersect a sphere in two circles. These circles intersect in two points, spaced 14 units apart, measured along the straight line connecting them. If the radii of the circles are 18 and 25 units, what is the radius of the sphere?

Comment: This interesting geometry problem was originally proposed for the American Invitational Mathematics Examination (AIME) by the late Professor Joseph Konhauser, a superb problemist and a great friend of Professor Berzsenyi.

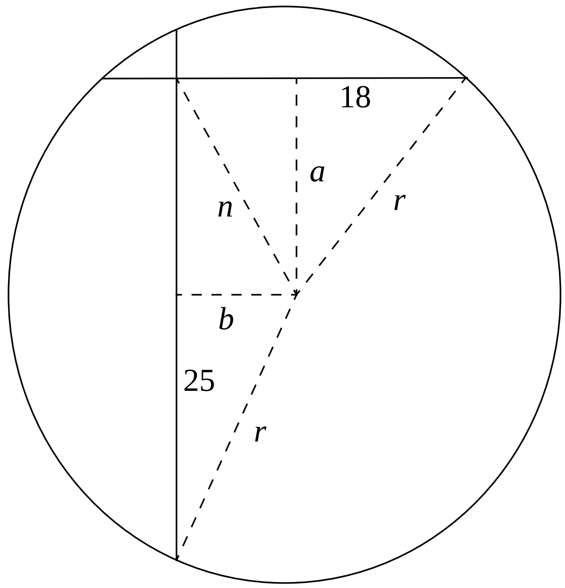
Solution 1 for 5/3/13 by Konstantin Getmanchuk (11/NY)

We have two perpendicular planes intersection a sphere. The intercepted section of a plane and a sphere is a circle.



Viewing the circle of radius 25 as a base, the height, x, of the center of the sphere above that base is the same as the height of the center of the circle of radius 18 above that base. $x = \sqrt{10^2 - 7^2} = \sqrt{275}$. As seen in the diagram, the radius, r, of the sphere is $r = \sqrt{x^2 + 25^2} = \sqrt{275 + 625} = 30$.

Solution 2 for 5/3/13 by Daniel Halpern-Leistner (11/PA)



This diagram is a cross-section perpendicular to both planes that intersect the sphere and coplanar with the centers of the two circles of the two circles formed by intersection and with the center of the sphere itself. First the values of a and b must be determined as functions of r via the Pythagorean theorem: $b^2 = r^2 - 25^2$ and $a^2 = r^2 - 18^2$. Again by the Pythagorean theorem, $n^2 = a^2 + b^2$. n is also the base of a triangle whose height is half the between the two intersections of the circles and whose hypotenuse is the radius of the sphere, because it connects the center with a point on the surface. Therefore, the Pythagorean relationship, $r^2 = n^2 + 7^2$, holds. Substituting one equation into the next: $n^2 = r^2 - 25^2 + r^2 - 18^2 = 2r^2 - 949$ and $r^2 = n^2 + 49 = 2r^2 - 900$, so $r^2 = 900$ and $r = 30$.

Solution 3 for 5/3/13 by Yan Zhang (11/VA)

We rotate the our coordinate system so that the *xy*-plane is parallel to the first plane intersecting the sphere, the *yz*-plane is parallel to the second plane, and the center of the sphere is at $(0, 0, 0)$. Clearly, the intersections are symmetric about the *xy*-plane, and their *z*-coordinates must be opposites of each other. So let their coordinates be $(x, y, 7)$ and $(x, y, -7)$.

Now all points on the first circle satisfy $x^2 + z^2 = 18^2$, and all the points on the second circle satisfy $y^2 + z^2 = 25^2$. Since $(x, y, 7)$ is on both circles, $x^2 + 7^2 = 18^2$ and $y^2 + 7^2 = 25^2$. The point $(x, y, 7)$ is on the sphere, whose radius is r, so

$$\begin{aligned} r^2 &= x^2 + y^2 + 7^2 \\ &= (18^2 - 7^2) + (25^2 - 7^2) + 7^2 \\ &= 900 \end{aligned}$$

and the radius is 30.