USA Mathematical Talent Search

PROBLEMS
Round 3 - Year 13 - Academic Year 2001-2002

1/3/13. We will say that a rearrangement of the letters of a word *has no fixed letters* if, when the rearrangement is placed directly below the word, no column has the same letter repeated. For instance, the blocks of letters below show that $ESARET$ is a rearrangement with no fixed letters of $TERESA$, but $REASTE$ is not.

$$\begin{matrix} T & E & R & E & S & A \\ E & S & A & R & E & T \end{matrix} \qquad\qquad \begin{matrix} T & E & R & E & S & A \\ R & E & A & S & T & E \end{matrix}$$

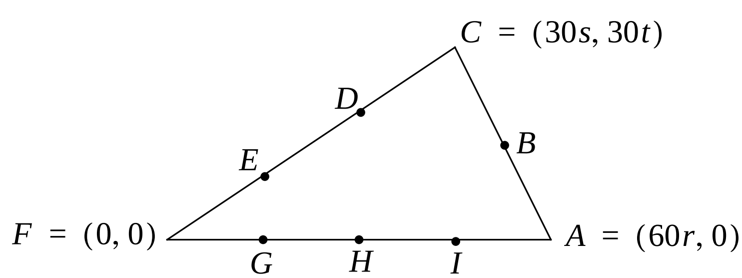
How many distinguishable rearrangements with no fixed letters does $TERESA$ have? (The two E's are considered identical.)

2/3/13. Without computer assistance, find five different sets of three positive integers $\{k, m, n\}$ such that $k < m < n$ and $\frac{1}{k} + \frac{1}{m} + \frac{1}{n} = \frac{19}{84}$.

[To think about, but not a part of the problem: How many solutions are there?]

3/3/13. Suppose $p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ is a monic polynomial with integer coefficients. [Here *monic* polynomial just means the coefficient of x^n is one.] If $(p(x))^2$ is a polynomial all of whose coefficients are non-negative, is it necessarily true that all the coefficients of $p(x)$ must be non-negative? Justify your answer.

4/3/13. As shown in the figure on the right, in ΔACF, B is the midpoint of $\overline{AC}$, D and E divide side $\overline{CF}$ into three equal parts, while G, H, and I divide side $\overline{FA}$ into four equal parts.



Seventeen segments are drawn to connect these six points to one another and to the opposite vertices of the triangle. Determine the points interior to ΔACF at which three or more of these line segments intersect one another.

To make grading easier, we have embedded the triangle into the first quadrant with point F at the origin, point C at $(30s, 30t)$, and point A at $(60r, 0)$, where r, s, and t are arbitrary positive real numbers. Please use this notation in your solutions.

5/3/13. Two perpendicular planes intersect a sphere in two circles. These circles intersect in two points, spaced 14 units apart, measured along the straight line connecting them. If the radii of the circles are 18 and 25 units, what is the radius of the sphere?

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Complete, well-written solutions to at least two of the problems above, accompanied by a **Cover Sheet,** should be mailed to

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and **postmarked no later than 6 January 2002**. Each participant is expected to develop solutions without help from others.