USA Mathematical Talent Search

PROBLEMS / SOLUTIONS / COMMENTS
Round 2 - Year 13 - Academic Year 2001-2002

Gene A. Berg, Editor

1/2/13. How many positive five-digit integers are there consisting of the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, in which one digit appears once and two digits appear twice? For example, 41174 is one such number, while 75355 is not.

Solution 1 by Diana Davis (11/NH):
The first step is to choose one digit to appear once: ${}_9C_1 = 9$.

Then choose the two digits that appear twice: ${}_8C_2 = 28$.

This yields the same result as choosing the two digits first: ${}_9C_2 = 36$,

and then choosing the digit to appear once: ${}_7C_1 = 7$,

because $9 \cdot 28 = 36 \cdot 7 = 252$.

The second step is to find the number of arrangements of five numbers (a, b, b, c, c).

This is $\dfrac{5!}{2!2!1!} = \dfrac{\text{(number of digits)}}{\text{(redundant arrangements)}} = 30$.

Then we multiply these numbers to find the total possibilities:

$252 \cdot 30 = 7560$ positive five digit integers with this property.

Solution 2 by Elana Ruse (10/NY): There are C_2^9 ways to choose the two digits that appear twice, and then 7 ways to choose the remaining digit. Then there are $C_2^9 \cdot 7 \cdot \dfrac{5!}{2!2!} = 7560$ such numbers, because there are $\dfrac{5!}{2!2!}$ distinct sequences of the 5 objects *A*, *A*, *B*, *B*, *C*, as in each sequence, the 2 *A*'s or the 2 *B*'s can be permuted among their positions, though the sequence really remains the same, in 2! ways.

Solution 3 by Simon Rubenstein-Salzedo (11/CA): First, we should count numbers of the form $aabbc$, where *a*, *b*, and *c* are distinct and from 1 to 9. There are $\dbinom{9}{2} \times 7$ of these, since *a* and *b* are interchangeable, and we do not want to over count by a factor of two. Next, there are $\dfrac{5!}{2!2!}$ ways of arranging *aabbc* into different orders. Multiplying these, we get the number of numbers in the form of the problem to be $\binom{9}{2} \times 7 \times \frac{5!}{2!2!} = 7560$ distinct numbers.

Editor's Comment: This problem was created by our Problem Editor, Dr. George Berzsenyi.

2/2/13. Determine, with proof, the positive integer whose square is exactly equal to the number

$$1 + \sum_{i=1}^{2001} (4i-2)^3.$$

[A computer solution will be worth at most 1 point.].

Solution by Jim Castelaz (12/TN): Recall that for summations from $i = 1$ to n:

$$\sum 1 = n$$
$$\sum i = (n)(n+1)/2$$
$$\sum i^2 = (n)(n+1)(2n-1)/6$$
$$\sum i^3 = [(n(n+1))/2]^2$$

Let all the following summations be from $i = 1$ to 2001

$$\begin{aligned} &1 + \sum (4i-2)^3 = \\ &= 1 + \sum (64i^3 - 96i^2 + 48i - 8) \\ &= 1 + 64 \cdot \sum i^3 - 96 \cdot \sum i^2 + 48 \cdot \sum i - 8 \cdot \sum 1 \\ &= 1 + 64\left(\frac{2001 \cdot 2002}{2}\right)^2 - 96\left(\frac{2001 \cdot 2002 \cdot 4001}{6}\right) + 48\left(\frac{2001 \cdot 2002}{2}\right) - (8 \cdot 2001) \\ &= 256512352096009 \end{aligned}$$

$$(256512352096009)^{1/2} = 16016003.$$

Editor's Comment: We are thankful to Dr. Jeremy Dover of NSA for proposing this nice problem.

3/2/13. Factor the expression

$$30(a^2+b^2+c^2+d^2)+68ab-75ac-156ad-61bc-100bd+87cd.$$

Solution 1 by Devin Averett (12/UT):
The factorization holds for all values of *a*, *b*, *c*, and *d*, so we can eliminate two variables at a time by setting them equal to zero, and then factor the resulting expression.

a & *b*: $30a^2+68ab+30b^2$

$= (5a+3b)(6a+10b)$ OR * $(3a+5b)(10a+6b)$

a & *c*: $30a^2-75ac+30c^2$

$= (6a-3c)(5a-10c)$ OR * $(3a-6c)(10a-5c)$

a & *d*: $30a^2-156ad+30d^2$

$= (10a-2d)(3a-15d)$ * OR $(2a-10d)(15a+-3d)$

$$\begin{array}{cc} (3a+5b) & (10a+6b) \\ \downarrow & \downarrow \\ \downarrow & \downarrow \\ \text{-}6c & \text{-}5c \\ \downarrow & \downarrow \\ \downarrow & \downarrow \\ \text{-}15d & \text{-}2d \end{array}$$

In each pair of alternate factorizations above, one of the alternates uses 3*a* and 10*a*. These are marked with *.

The resulting factorization is

$$(3a+5b-6c-15d)(10a+6b-5c-2d)$$

Solution 2 by Nataliya Ostrovskaya (11/NY): Assume the given expression factors as

$$(Aa+Bb+Cc+Dd)(Ea+Fb+Gc+Hd).$$

Since the coefficients of a^2, b^2, c^2 and d^2 are all 30, $E = \frac{30}{A}$, $F = \frac{30}{B}$, $G = \frac{30}{C}$, and $H = \frac{30}{D}$.
The expression becomes

$$(Aa+Bb+Cc+Dd)\left(\frac{30}{A}a+\frac{30}{B}b+\frac{30}{C}c+\frac{30}{D}d\right).$$

Since we get the *ab* term by multiplying the *a* term from the first factor with the *b* term of the second, or the *b* term of the first by the *a* term of the second, the coefficient of *ab* in the expansion is $30\frac{A}{B}+30\frac{B}{A} = 68$. Let $r = \frac{A}{B}$. Then $30r^2-68r+30 = 0$, so $\frac{A}{B} = \frac{5}{3}$ or $\frac{3}{5}$. Similarly, we get $\frac{A}{C} = -2$ or $\frac{-1}{2}$; $\frac{A}{D} = \frac{-1}{5}$ or -5; $\frac{B}{C} = \frac{-6}{5}$ or $\frac{-5}{6}$; $\frac{B}{D} = -3$ or $\frac{-1}{3}$; $\frac{C}{D} = \frac{2}{5}$ or $\frac{5}{2}$.

Now,

$$\left(-2 \text{ or } \frac{-1}{2}\right) = \frac{A}{C} = \left(\frac{A}{B}\right)\left(\frac{B}{C}\right) = \left(\frac{5}{3} \text{ or } \frac{3}{5}\right)\left(\frac{-6}{5} \text{ or } \frac{-5}{6}\right)$$

implies, by inspection, that $\left(\frac{A}{B}, \frac{B}{C}\right) = \left(\frac{3}{5}, \frac{-5}{6}\right)$ or $\left(\frac{5}{3}, \frac{-6}{5}\right)$.

Similarly, since

$$\left(\frac{5}{3} \text{ or } \frac{3}{5}\right) = \frac{B}{A} = \left(\frac{B}{C}\right)\left(\frac{C}{A}\right) = \left(\frac{-6}{5} \text{ or } \frac{-5}{6}\right)\left(-2 \text{ or } \frac{-1}{2}\right)$$

we must have $\left(\frac{B}{C}, \frac{C}{A}\right) = \left(\frac{-6}{5}, \frac{-1}{2}\right)$ or $\left(\frac{5}{6}, -2\right)$.

Finally,

$$\left(\frac{2}{5} \text{ or } \frac{5}{2}\right) = \frac{C}{D} = \left(\frac{C}{A}\right)\left(\frac{A}{D}\right) = \left(-2 \text{ or } \frac{-1}{2}\right)\left(\frac{-1}{5} \text{ or } -5\right)$$

implies that $\left(\frac{C}{A}, \frac{A}{D}\right) = \left(-2, \frac{-1}{5}\right)$ or $\left(\frac{-1}{2}, -5\right)$.

If $\frac{A}{B} = \frac{3}{5}$, then $\left(\frac{A}{B}, \frac{B}{C}, \frac{C}{A}, \frac{A}{D}\right) = \left(\frac{3}{5}, \frac{-5}{6}, -2, \frac{-1}{5}\right)$, and so $(A, B, C, D) = \left(A, \frac{5A}{3}, -2A, 5A\right)$. It is easy to check that all the pair-wise ratios of these numbers are indeed the ratios we found above.

Letting $A = 3$, we get that the expression factors as

$$(3a + 5b - 6c - 15d)(10a + 6b - 5c - 2d).$$

Editor's Comment: We are indebted to Professor Harold Reiter of the University of North Carolina in Charlotte, NC, for this interesting problem. Dr. Reiter is a staunch friend and supporter of the USAMTS program.

4/2/13. Let $X = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9)$ be a 9-long vector of integers. Determine X if the following seven vectors were all obtained from X by deleting three of its components: $Y_1 = (0, 0, 0, 1, 0, 1)$, $Y_2 = (0, 0, 1, 1, 1, 0)$, $Y_3 = (0, 1, 0, 1, 0, 1)$, $Y_4 = (1, 0, 0, 0, 1, 1)$, $Y_5 = (1, 0, 1, 1, 1, 1)$, $Y_6 = (1, 1, 1, 1, 0, 1)$, $Y_7 = (1, 1, 0, 1, 1, 0)$.

Solution 1 by Jeffrey Kuan (8/IL): From Y_1, we know that at least four components of X are 0. From Y_5 and Y_6 we know that at least five components of X are 1. So exactly five components are 1 and exactly four components are 0. Looking at Y_1, one can deduce that $x_9 = 1$. Why? Well, at least one 1 occurs after the fourth (and last) 0. Observing Y_6, at least one 0 occurs between the last and second-to-last 1. So, $x_8 = 0$. Again, looking at Y_1, at least one 1 occurs between the third and fourth 0. Since x_8 is the fourth 0, $x_7 \neq 0$. x_7 must equal 1. Continuing in a similar manner you find that

$$\text{the answer is } X = (1, 0, 0, 1, 0, 1, 1, 0, 1).$$

Solution 2 by Tamara Broderick (11/OH): $X = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9)$. In Y_5, five 1 terms occur, and in Y_1, four 0 terms occur. Since these terms must be in the original vector *X*, vector *X* must have four 0 terms and five 1 terms.

Then, since Y_1 already contains all the zero terms, no 0 terms have been deleted from it. Because its last term is 1, no 0 terms could have occurred in *X* after this 1, and $x_9 = 1$. Y_6 already contains all of the 1 terms (i.e. none have been omitted), and the last 1 shown is x_9, so $x_8 = 0$. Again Y_1 is not missing a 0 so x_8 and x_9 are represented by the final 0 and 1, and x_7 must be a 1. Next, looking at Y_4, we know that the two final digits of *X* are not both 1; and 0 must therefore come between or after the last two 1's in Y_4. With this 0, all the 0 terms of *X* are represented and we know $x_1 = 1$. Likewise, all of the 1 terms are shown in Y_5 so $x_2 = 0$. In Y_2 the 1's on both ends are missing. When supplied, the total number of 1's is represented and x_3 must equal 0. Finally, Y_7 is obviously missing the final 1 and the 0's of the x_2 and x_3 terms. Substituting these in, we have the whole 9-term vector

$$X = (1, 0, 0, 1, 0, 1, 1, 0, 1).$$

Editor's comment: This problem was proposed by Dr. Gene Berg of NSA on the basis of an article by Vladimer I. Levenshtein, published in the January 2001 issue of the *IEEE Transactions of Information Theory.* The article answers several questions such as, "What is the minimum number *m* of distinct binary vectors *Y* of length $n - k$ from which one can reconstruct any *n*-dimensional vector?"

5/2/13. Let R and S be points on the sides $\overline{BC}$ and $\overline{AC}$, respectively, of ΔABC, and let P be the intersection of $\overline{AR}$ and $\overline{BS}$. Determine the area of ΔABC if the areas of ΔAPS, ΔAPB, and ΔBPR are 5, 6, and 7, respectively.

Solution by Connie Yee (10/NY): When two adjacent triangles share a height, the ratio of their areas is equal to the ratio of their bases. For ΔASP and ΔABP, whose common height is AM

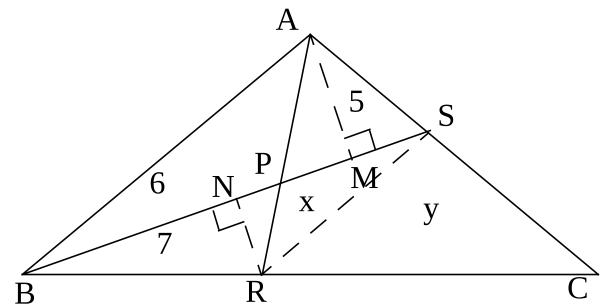
$$\frac{SP}{PB} = \frac{5}{6}.$$

For ΔSPR and ΔBPR, whose common height is RN

$$\frac{SP}{PB} = \frac{x}{7}.$$

Therefore $x = 35/6$.

Not to scale.



Similarly, for ΔBSR and ΔRSC, $\quad \dfrac{BR}{RC} = \dfrac{7+x}{y}$

and for ΔABR and ΔACR, $\quad \dfrac{BR}{RC} = \dfrac{7+x}{y}$.

Therefore $\dfrac{7+x}{y} = \dfrac{13}{5+x+y}$, and $y = \dfrac{(7+x)(5+x)}{6-x} = \dfrac{5005}{6}$.

So $$\text{Area}(\Delta ABC) = 5+6+7+x+y = 5+6+7+\frac{35}{6}+\frac{5005}{6}$$
$$= 858.$$

Editor's comments: We thank former USAMTS participant Surat Intasang for proposing this problem, inspired by Problem 6 of the 1985 American Invitational Mathematics Examination. We hereby wish Surat success with his efforts to transplant the USAMTS to his native Thailand.