U S A Mathematical Talent Search

PROBLEMS

Round 2 - Year 12 - Academic Year 2000-2001

1/2/12. A well-known test for divisibility by 19 is as follows: Remove the last digit of the number, add twice that digit to the truncated number, and keep repeating this procedure until a number less than 20 is obtained. Then, the original number is divisible by 19 if and only if the final number is 19. The method is exemplified on the right; it is easy to check that indeed 67944 is divisible by 19, while 44976 is not.

$$\begin{array}{cccccc}
 & 6 & 7 & 9 & 4 & \cancel{4} \\
 & & & & 8 & \\ \hline
 & 6 & 8 & 0 & \cancel{2} & \\
 & & & 4 & & \\ \hline
 & 6 & 8 & \cancel{4} & & \\
 & & 8 & & & \\ \hline
 & 7 & \cancel{6} & & & \\
1 & 2 & & & & \\ \hline
1 & 9 & & & &
\end{array}
\qquad\qquad
\begin{array}{cccccc}
 & 4 & 4 & 9 & 7 & \cancel{6} \\
 & & & 1 & 2 & \\ \hline
 & 4 & 5 & 0 & \cancel{9} & \\
 & & 1 & 8 & & \\ \hline
 & 4 & 6 & \cancel{8} & & \\
 & 1 & 6 & & & \\ \hline
 & 6 & \cancel{2} & & & \\
 & 4 & & & & \\ \hline
1 & 0 & & & &
\end{array}$$

Find and prove a similar test for divisibility by 29.

2/2/12. Compute $1776^{1492!} \pmod{2000}$; i.e., the remainder when $1776^{1492!}$ is divided by 2000. (As usual, the exclamation point denotes factorial.)

3/2/12. Given the arithmetic progression of integers

$$308,\ 973,\ 1638,\ 2303,\ 2968,\ 3633,\ 4298\,,$$

determine the unique geometric progression of integers,

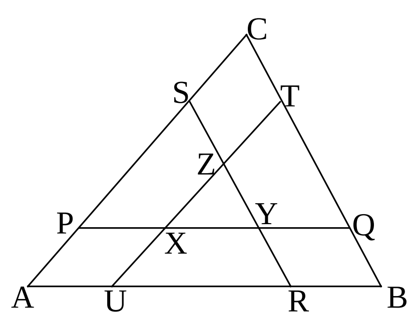
$$b_1, b_2, b_3, b_4, b_5, b_6$$

so that

$$308 < b_1 < 973 < b_2 < 1638 < b_3 < 2303 < b_4 < 2968 < b_5 < 3633 < b_6 < 4298\,.$$

4/2/12. Prove that every polyhedron has two vertices at which the same number of edges meet. [*Editor's comment:* Proofs are difficult to grade. Please try to keep your solution short and well written.]

5/2/12. In ΔABC, segments *PQ*, *RS*, and *TU* are parallel to sides *AB*, *BC*, and *CA*, respectively, and intersect at the points *X*, *Y*, and *Z*, as shown in the figure on the right.



Determine the area of ΔABC if each of the segments *PQ*, *RS*, and *TU* bisects (halves) the area of ΔABC, and if the area of ΔXYZ is one unit. Your answer should be in the form $a + b\sqrt{2}$, where *a* and *b* are positive integers.

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Complete, well-written solutions to at least two of the problems above, accompanied by a completed **Cover Sheet** (available on the web site http://www.nsa.gov/programs/mepp/usamts.html), should be sent to the following address and **postmarked no later than November 13, 2000**. Also include an **Entry Form** if you did not participate in Round 1. Each participant is expected to develop solutions without help from others.

USA Mathematical Talent Search
COMAP Inc., Suite 210
57 Bedford Street
Lexington, MA 02420-4350