USA Mathematical Talent Search

PROBLEMS / SOLUTIONS / COMMENTS
Round 1 - Year 12 - Academic Year 2000-2001

Gene A. Berg, Editor

1/1/12. Determine the smallest five-digit positive integer N such that $2N$ is also a five-digit integer and all ten digits from 0 to 9 are found in N and $2N$.

Solution 1 by Alex Lang (8/WI):
First step: Assign variables to the digits. N is assigned the variables $ABCDE$ and $2N$ is assigned the variables $FGHIJ$.

Second step: Solve for N and $2N$. A cannot equal 0 since N would then be considered a 4 digit number and to satisfy the conditions N needs to be a 5 digit number. The next smallest number that A could equal is 1. Therefore, A will be assumed equal to 1 until all possible solutions with A = 1 have been proved false. $F = 2$ since $2A = 2(1) = 2$. B will be assumed equal to 3 since that is the smallest number not already used. C will be assumed equal to 4. Then $G = 6$ since $2B = 2(3) = 6$. D will be assumed to equal 5. H = 9 since 2C + carry over from 2D = 2(4) + 1 = 9. But, if E = 8, $2(D)$ + carry over = 2(5) + 1 = 11, so $I = 1$. That is a contradiction because A also equals 1 and each number is supposed to be used once. Also, if $E = 0$, $2E$ would also equal 0, and another contradiction would occur. Therefore, $D \neq 5$. D will then be assumed equal to 7. That is because G already equals 6, therefore 7 is the next smallest number. Then $H = 9$ since $2C$ + carryover = 2(4) + 1 = 9. Then E would have to equal 8 because if $E = 0$ then $2E$ would also equal 0 and a contradiction would arise. But, if $E = 8$ then $2E = 2(8) = 16$ and therefore $J = 6$, another contradiction occurs since G already equals 6. Therefore, $D \neq 7$. D will then be assumed to equal 8. Then $H = 9$ since $2C$ + carry over = 2(4) + 1 = 9. E will then be assumed to equal 5. Then $2E$ = 2(5) = 10, but J would only equal the last digit which is 0, I would then equal $2D$ + carry over = 2(8) + 1 = 17, so $I = 7$. That solution satisfies the specified conditions.

Therefore, ***N* = 13,485** and $2N$ = 26,970.

Solution 2 by Valerie Lee (10/NY):

□ N is comprised of digits $ABCDE$, while $2N$ is $FGHIJ$.

1. Let $A = 1$.
2. If A is 1, then $B \neq 2$ because F would equal 2, and $B \neq 2$ because if $C < 5$, then $G = 0$ and C = 5-9, then $G = 1$. Let $B = 3$.
3. $C \neq$ 0-3, so let $C = 4$.

□ Thus far we have $134DE \bullet 2 = 26[8, 9]IJ$.

4. a) $D \neq$ 0-4, so let $D = 5$, so we would have $1345E \bullet 2 = 269[0, 1]J$ and there would be no way to conclude $J = 7$ or 8, so $D \neq 5$.
   b) Let $D = 6$, so $1346E \bullet 2 = 269[2, 3]J$, but 6 is used twice, so $D \neq 6$.

c) Let $D = 7$, so $1347E \bullet 2 = 269[4, 5]J$, but there is no answer for E so that $J = 0$ or 8, so $D \neq 7$.

d) Let $D = 8$, so $1348E \bullet 2 = 269[6, 7]J$. Now $E = 0$ or 5. $E = 0$ gives contradiction. So let $E = 5$. $13485 \bullet 2 = 26970$. This uses all ten digits.

Answer: *N* = 13485.

Solution 3 by Paul Wrayno (11/NC): Let $N = ABCDE$, $2N = FGHIJ$.
To be smallest, ideally $A = 1$, and $F = 2$, $B = 3$, $G = 6$, if it is possible to create N and $2N$ with these values, because they are the absolute least values for the first two digits. The lowest remaining digit is 4, so ideally $C = 4$, causing $H = 9$ because the remaining digits guarantee a carry from the $2 \bullet D$. This leaves 0, 5, 7, and 8. Since 0 can only be achieved by $2 \bullet 5$ without a carry, $E = 5$ and $J = 0$. $8 \bullet 2 + 1 = 17$, which fits the other four digits, so $D = 8$ and $J = 7$. This is the only (N, $2N$) pair that has the ideal first three digits, so $N = 13485$ is the smallest.
Answer: *N* = 13485.

Editor's Comment: We are indebted to Dr. Béla Bajnok of Gettysburg College for posing this problem and for his continuing assistance with USAMTS. Dr. Bajnok claims the next smallest solutions (after 13485) are 13548, 13845, and 14538. He notes they all use the same five digits.

2/1/12. It was recently shown that $2^{2^{24}} + 1$ is not a prime number. Find the four rightmost digits of this number.

Solution 1 by Christopher Lyons (12/CA): This problem is equivalent to asking, "What is the remainder when $2^{2^{24}} + 1$ is divided by 10,000?" Since we are only concerned with the remainder, I chose to use the mathematical tool that is all about remainders: the congruence. My strategy in attacking this problem is to use the rules of congruences to build up from
$2^{2^1} = 4 \equiv 4 (\text{mod } 10{,}000)$ to $2^{2^{24}} = 7536 (\text{mod } 10{,}000)$, and to add 1 to get
$2^{2^{24}} + 1 = 7537 (\text{mod } 10{,}000)$.

We start by evaluating the congruence of 2^{2^1} modulo 10,000. In fact, throughout the description of this problem, all congruences will be modulo 10,000.

$$2^{2^1} = 4 \equiv 4 (\text{mod } 10{,}000)$$

Since we can multiply congruences just like equations, we will multiply this congruence by itself. But first we must show an identity that will be useful for the rest of the problem:

$$2^{2^n} \times 2^{2^n} = 2^{(2^n + 2^n)} = 2^{2(2^n)} = 2^{2^{n+1}}$$

Thus, when we multiply 2^{2^n} by 2^{2^n}, we get $2^{2^{n+1}}$. For example $2^{2^6} \times 2^{2^6} = 2^{2^7}$. We can apply this identity repeatedly with congruences to get all the way from 2^{2^1} to $2^{2^{24}}$. Let us write out the first few steps:

$$2^{2^2} = 2^{2^1} \times 2^{2^1} = 4 \times 4 = 16$$

$$2^{2^2} = 16$$

$$2^{2^3} = 2^{2^2} \times 2^{2^2} = 16 \times 16 = 256$$

$$2^{2^3} = 256$$

$$2^{2^4} = 2^{2^3} \times 2^{2^3} = 256 \times 256 = 65536 \equiv 5536 (\text{mod } 10{,}000)$$

$$2^{2^4} \equiv 5536$$

$$2^{2^5} = 2^{2^4} \times 2^{2^4} = 65536 \times 65536 = 30647296 \equiv 7296$$

$$2^{2^5} \equiv 7296$$

We continue this process, obtaining the congruences of 2^{2^6}, 2^{2^7}, ..., $2^{2^{24}}$. The following is a table of all these congruences. (I used an eight-digit hand held calculator to find these values, since this was the easiest way for me.)

$2^{2^1} \equiv 4$	$2^{2^9} \equiv 4096$	$2^{2^{17}} \equiv 3696$
$2^{2^2} \equiv 16$	$2^{2^{10}} \equiv 7216$	$2^{2^{18}} \equiv 416$
$2^{2^3} \equiv 256$	$2^{2^{11}} \equiv 656$	$2^{2^{19}} \equiv 3056$
$2^{2^4} \equiv 5536$	$2^{2^{12}} \equiv 336$	$2^{2^{20}} \equiv 9136$
$2^{2^5} \equiv 7296$	$2^{2^{13}} \equiv 2896$	$2^{2^{21}} \equiv 6496$
$2^{2^6} \equiv 1616$	$2^{2^{14}} \equiv 6816$	$2^{2^{22}} \equiv 8016$
$2^{2^7} \equiv 1456$	$2^{2^{15}} \equiv 7856$	$2^{2^{23}} \equiv 6256$
$2^{2^8} \equiv 9936$	$2^{2^{16}} \equiv 6736$	$2^{2^{24}} \equiv 7536$

Now that we have obtained this last congruence, we may add it to another congruence to produce the desired result:

$$2^{2^{24}} + 1 \equiv 7536 + 1 \equiv 7537 \pmod{10{,}000}.$$

This final congruence is equivalent to writing

$$2^{2^{24}} + 1 = 10000 \times K + 7537$$

where K is some positive integer (a large one, probably!). Looking at the right side of this equation, we see that the multiple of 10,000 will have (at least) four trailing zeros. When we add the multiple and its four trailing zeros to 7537, we get a number whose four rightmost digits are 7537.

The four rightmost digits of $2^{2^{24}} + 1$ are 7537.

Solution 2 by Aleksandr Kivenson (10/NY): To begin, it is necessary to declare that to obtain the four rightmost digits of a product, only the four rightmost digits of the factors need to be multiplied (for example, to get the four rightmost digits of the product of 12,345 and 67,890, all you have to do is multiply 2,345 by 7,890 and take the four rightmost terms of the result). Since my approach to this problem is to use known large numbers and multiply their four rightmost digits to get the answer, I will use this method.

Since when you multiply numbers which have the same base but different powers you add the powers (for example $2^2 \times 2^3 = 2^{2+3}$), it would be easy to find numbers whose rightmost four digits I should multiply by looking for large numbers expressed as powers of 2, finding the ones whose powers add up to 2^{24}, and then multiplying the rightmost 4 digits of these numbers to get the rightmost four digits of $2^{2^{24}}$.

Conveniently, such a list of known powers of two exists in the form of Mersenne prime numbers. These are a special type of prime numbers that are expressed as $2^x - 1$. Many such prime numbers are known and I used a web page

http://www.isthe.com/chongo/tech/math/prime/mersenne.html

to find numerical values for these primes. I chose the powers of 2 whose powers added up to the exponent $2^{24} = 16777216$. I ignored the -1 because when I chose the powers I would use I obtained the last four digits of their numerical value, added one to each, and then multiplied them out to get the last four digits of $2^{2^{24}}$. The powers which I used and their numerical value are as follows:

:

Power of 2	Last four digits of that power
2^{12}	4096
2^{607}	8128
2^{607}	8128
2^{19937}	1472
2^{44497}	8672
2^{216091}	8448
2^{216091}	8448
$2^{1398269}$	5712
$2^{2976221}$	1152
$2^{2976221}$	1152
$2^{2976221}$	1152
$2^{2976221}$	1152
$2^{2976221}$	1152
Sums of powers: 16,777,216	Last four digits of above values: 7,536

From multiplying the last four digits of the numerical value of each power, I obtained a number whose last four digits were 7,536. This means the last four digits of $2^{2^{24}}$ are 7,536.

Therefore, the last four digits of $2^{2^{24}} + 1$ are 7,537.

Solution 3 by Zhihao Liu (11/IL):
Answer: **7537**
Proof: [See the Editor's Comment below for a quick review of some of these terms and concepts.]

Note that $2^{24} \equiv (2^9)^2(64) \equiv 144 \times 64 = 9216 \equiv 216(\text{mod } 500)$. By Euler's Theorem,

$2^{2^{24}} \equiv 2^x (\bmod 5^{\,4})$, if $2^{24} \equiv x (\bmod \phi(5^4))$. Since $\phi(625) = 500$, Euler's Theorem says $2^{2^{24}} \equiv 2^x (\bmod 625)$, if $2^{24} \equiv x (\bmod 500)$. It follows that $2^{2^{24}} \equiv 2^{216} (\bmod 625)$. By doubling for a while we find $2^{27} = 134217728 \equiv 228 (\bmod 625)$, and so $2^{216} \equiv 228^8 \equiv (228^2)^4 \equiv 109^4 \equiv (109^2)^2 \equiv 6^2 \equiv 36 (\bmod 625)$. Since 2^4 and 5^4 are relatively prime, by the Chinese Remainder Theorem there is a unique n, $0 \le n < 2^4 \cdot 2^5 = 10000$ that satisfies both $n \equiv 36 (\bmod 625)$, and $n \equiv 0 (\bmod 16)$. Note that $n = 7536$ satisfies both of these congruences, and $2^{2^{24}} \equiv n (\bmod 10000)$. Therefore the last four digits of $2^{2^{24}} + 1$ are **7537**.

Solution 4 by Jacob Licht (11/CT):

Euler's Theorem states: If $(a,m) = 1$, then $a^{\phi(m)} \equiv 1 (\bmod m)$. If $a = 2$ and $m = 625$, $\phi(625) = \phi(5^4) = 5^3(5-1) = 500$. By the Division Algorithm $\exists \quad q, r \quad \in Z \quad \ni$ $2^{24} = 500q + r$ and $0 \le r < 500$. [Read: By the Division Algorithm there exist q and r, elements of the integers Z, such that $2^{24} = 500q + r$ and $0 \le r < 500$.] Since $2^{10} = 1024 \equiv 24 (\bmod 500)$, $\Rightarrow 2^{24} = (2^{10})^2 2^4 \equiv (24^2) 16 \equiv 216 (\bmod 500)$. So $r = 216$, and $2^{2^{24}} = 2^{500q + 216} = (2^{500})^q 2^{216} \equiv (1)^q 2^{216} \equiv 2^{216} (\bmod 625)$. Note that $2^8 = 256$, so

$$2^{16} = 256^2 = 65536 \equiv 536 (\bmod 625)$$
$$2^{32} = 536^2 \equiv 421 (\bmod 625)$$
$$2^{64} = 421^2 \equiv 366 (\bmod 625)$$
$$2^{128} = 366^2 \equiv 206 (\bmod 625).$$

So, $2^{216} = 2^{128 + 64 + 16 + 8} \equiv (206)(366)(536)(256) \equiv 36 (\bmod 625)$.

Thus $2^{2^{24}} \equiv 36 (\bmod 635)$, and since $2^4 = 16$ divides $2^{2^{24}}$ so $2^{2^{24}} \equiv 0 (\bmod 16)$. We also have that $625 \equiv 1 (\bmod 16)$ and $36 \equiv 4 (\bmod 16)$, so $12(625) + 36 \equiv 0 (\bmod 16)$. Now by the Chinese Remainder Theorem $2^{2^{24}} \equiv 12(625) + 36 \equiv 7536 (\bmod 10000)$.

So the four rightmost digits of $2^{2^{24}} + 1$ **are 7537**.

Editor's Comment: In what he described as "the deepest computation in history whose result was a simple yes/no answer," Richard Crandall of the Center for Advanced Computation at Reed College, together with Ernest Myer, formerly of Case Western Reserve University, and Jason Papadopoulos of the University of Maryland, have verified that the 24th Fermat number, $2^{2^{24}} + 1$, is not a prime number. For more information about the 24th Fermat number, visit the web site www.perfsci.com/. This problem was created by Gene Berg of NSA.

The solutions to these problems give us an opportunity to briefly introduce *Euler's function* $\phi(m)$*, Euler's generalization of Fermat's Theorem,* and *the Chinese Remainder Theroem*. Our goal is to introduce some of the notation and terminology of this subject to young mathematicians seeing this for the first time, and possibly help them understand the proofs. For more details see (a) *An Introduction to the Theory of Numbers* by G. H. Hardy and E. M. Wright, published by Clarendon Press, or (b) *The Art of Problem Solving, Volumes 1 and 2*, by Richard Rusczyk and Sandor Lehoczky, published by Greater Testing Concepts, P.O. Box 5014, New York, NY 10185-5014.

For a brief discussion of *Congruences, Fermat's Theorem*, and *the Extended Euclidean Algorithm (EEA) for finding Greatest Common Divisors* see the Solutions to Problem 1/2/11 from Year 11 of the USAMTS.

Definition (Euler's function $\phi(m)$*)*: For an integer *m*, let $\phi(m)$ denote the number of positive integers less than *m* and relatively prime to *m*. For example, consider *m* = 20: there are eight positive integers less than 20 which are relatively prime to 20, namely 1, 3, 7, 9, 11, 13, 17, and 19, so $\phi(20) = 8$. Since $m = 17$ is prime, all sixteen positive integers less than 17 are prime to 17, and $\phi(17) = 16$. If p is prime, then $\phi(p^k) = p^k\left(1-\frac{1}{p}\right)$. If *m* and *n* are relatively prime integers, then $\phi(mn) = \phi(m)\phi(n)$. For example:

$$\phi(20) = \phi(4\times 5) = \phi(4)\times\phi(5) = \left[4\left(1-\frac{1}{2}\right)\right]\times[4] = 8$$

Theorem (Euler's generalization of Fermat's Theorem): If *a* and *m* are integers with Greatest Common Divisor GCD(*a*, *m*) = 1, then

$$a^{\phi(m)} \equiv 1 (\text{mod } m).$$

For example, if *a* is any integer relatively prime to 20 [i.e. $a \in \{1, 3, 7, 9, 11, 13, 17, 19\}$], then $a^8 \equiv 1 (\text{mod } 20)$.

Chinese Remainder Theroem: If $m_1, m_2, \ldots, m_k$ are positive integers that are pairwise relatively prime [that is, GCD(m_i, m_j) = 1 for $1 \le i < j \le k$], then for any integers $a_1, a_2, \ldots, a_k$ the system of congruences $y \equiv a_i (\text{mod } m_i), i = 1, 2, \ldots, k$, has a simultaneous solution *y* that is uniquely determined modulo $m = m_1 m_2 \cdots m_k$. [A similar theorem applies to polynomials.]

As an example, find an integer *c* with $0 \le c < 3 \cdot 7 \cdot 11 \cdot 13 = 3003$ such that

$$c \equiv 2 (\text{mod } 3),$$
$$c \equiv 4 (\text{mod } 7),$$

$$c \equiv 6 \pmod{11}, \text{ and}$$
$$c \equiv 8 \pmod{13}.$$

Solution: We do this in three steps, solving for the first two equations, then for the first three equations, and finally for all four equations. In each step we use the Extended Euclidean Algorithm (EEA) (for examples of this algorithm in use, see the Solution to Problem 2/1/11 of Year 11).

Step 1. Find x satisfying $x \equiv 2 \pmod 3 \equiv a_1 \pmod{m_1}$ and $x \equiv 4 \pmod 7 \equiv a_2 \pmod{m_2}$. $\mathrm{GCD}(m_1, m_2) = 1$ so by the EEA there are integers f_1 and f_2 so that $1 \equiv f_1 m_1 + f_2 m_2 \equiv f_1 \cdot 3 + f_2 \cdot 7$. The EEA finds $f_1 = 5$ and $f_2 = -2$. Now choose

$$x = a_1 + (a_2 - a_1) \cdot f_1 \cdot m_1 = 2 + (4-2) \cdot 5 \cdot 3 = 32.$$

Observe x satisfies the requirements of step 1.

Step 2. Find y satisfying $y \equiv 2 \pmod 3 \equiv a_1 \pmod{m_1}$, $y \equiv 4 \pmod 7 \equiv a_2 \pmod{m_2}$, and $y \equiv 6 \pmod{11} \equiv a_3 \pmod{m_3}$. $\mathrm{GCD}(m_1 m_2, m_3) = 1$ so by the EEA there are integers g_1 and g_2 so that $1 \equiv g_1 m_1 m_2 + g_2 m_3 \equiv g_1 \cdot 21 + g_2 \cdot 11 \equiv (-1) \cdot 21 + 2 \cdot 11$. Now choose

$$y = a_1 + (a_2 - a_1) \cdot f_1 \cdot m_1 + (a_3 - x) \cdot g_1 \cdot m_1 \cdot m_2 = 32 + (6 - 32)(-1)(3)(7) = 578.$$

where

$$x = a_1 + (a_2 - a_1) \cdot f_1 \cdot m_1.$$

Observe $y = 578$ satisfies step 2.

Step 3. Find c satisfying $c \equiv 2 \pmod 3 \equiv a_1 \pmod{m_1}$, $c \equiv 4 \pmod 7 \equiv a_2 \pmod{m_2}$, $c \equiv 6 \pmod{11} \equiv a_3 \pmod{m_3}$ and $c \equiv 8 \pmod{13}$. $\mathrm{GCD}(m_1 m_2 m_3, m_4) = 1$ so by the EEA there are integers h_1 and h_2 so that

$1 \equiv h_1 m_1 m_2 m_3 + h_2 m_4 \equiv h_1 \cdot 3 \cdot 7 \cdot 11 + h_2 \cdot 13 \equiv 4 \cdot 231 + (-71) \cdot 13$. Now choose

$$\begin{aligned} z &= a_1 + (a_2 - a_1) \cdot f_1 \cdot m_1 + (a_3 - x) \cdot g_1 \cdot m_1 \cdot m_2 + (a_4 - y) \cdot h_1 \cdot m_1 \cdot m_2 \cdot m_3 \\ &= y + (a_4 - y) \cdot h_1 \cdot m_1 \cdot m_2 \cdot m_3. \end{aligned}$$

where

$$x = a_1 + (a_2 - a_1) \cdot f_1 \cdot m_1 \text{ and } y = a_1 + (a_2 - a_1) \cdot f_1 \cdot m_1 + (a_3 - x) \cdot g_1 \cdot m_1 \cdot m_2.$$

Observe $z = 578 + (8 - 578) \cdot 4 \cdot 3 \cdot 7 \cdot 11 = -526680$, and reduced to $c = 2426 \equiv z \pmod{3003}$ satisfies step 3 and the original requirement.

3/1/12. Determine the integers a, b, c, d, and e for which

$$(x^2 + ax + b)(x^3 + cx^2 + dx + e) = x^5 - 9x - 27.$$

Solution 1 by Christopher Church (10/KY): This problem essentially asks us to factor $x^5 - 9x - 27$ into a quadratic and a cubic polynomial. Theoretically, a TI-89 graphing calculator could factor this polynomial. I, however, do not have access to the TI-89. However, I did use the TI-85 to calculate the roots of the polynomial. My calculator returned the five values shown here:

$$2.1541,\ 4229 \pm 1.9998i,\ -1.5 \pm 0.8\dot{6}60i$$

The last one, based on my knowledge of the quadratic formula, was $-\frac{3}{2} \pm \sqrt{\frac{3}{2}}i$. This I could transfer into a polynomial of degree two. The result, using the fact that x equals the values above, is $x^2 + 3x + 3$. Using long division of polynomials, I found that

$$(x^2 + 3x + 3)(x^3 - 3x^2 + 6x - 9) = x^5 - 9x - 27.$$

Thus the integers that the problem asked for are found here. They are $a = 3$, $b = 3$, $c = -3$, $d = 6$, and $e = -9$.

Solution 2 by Laura Pruitt (11/MA): Multiply out the left side of the equation to get

$$x^5 + (a + c)x^4 + (ac + b + d)x^3 + (ad + bc + e)x^2 + (ae + bd)x + be = x^5 - 9x - 27$$

Comparing coefficients in this new equation, it is easy to find equations relating the variables *a, b, c, d,* and *e*:

1. $a + c = 0$
2. $ac + b + d = 0$
3. $ad + bc + e = 0$
4. $ae + bd = -9$
5. $be = -27$

Initial observations on these equations:

(i) Simple substitutions: $a = -c$, or $c = -a$, $b = -27/e$ or $e = -27/b$.

(ii) Since all variables are integers, there are only eight possibilities for

(b,e): (1,–2), (–1,27), (27,–1), (–2 ,1), (3,–9), (–3,9), (9,–3), (–9,3).

This leaves us with three equations (2, 3, and 4) and one variable (*d*) that we have not used yet. Substitute $-c$ for *a* and solve 2, 3, and 4 for *d* in terms of *a, b, c,* and *e*:

2. $-c^2 + b + d = 0$ -> $d = c^2 - b$
3. $-cd + bc + e = 0$ -> $d = (bc + e)/c$
4. $-ce + bd = -9$ -> $d = (ce - 9)/b$

Therefore $c^2 - b = (bc + e)/c = (ce - 9)/b$.

In pairs, solve for c in terms of b and e:

Comb. 1 (use 2 and 3)	Comb. 2 (use 2 and 4)	Comb. 3 (use 3 and 4)
$c^2 - b = (bc+e)/c$	$c^2 - b = (ce-9)/b$	$(bc+e)/c = (ce-9)/b$
$c^3 - 2bc - e = 0$	$bc^2 - ec + (9-b^2) = 0$	$ec^2 - (b^2+9)c - be = 0$
discard: cubic	$c = \dfrac{[e \pm \sqrt{e^2 - 4b(9-b^2)}]}{2b}$	$c = \dfrac{\left[b^2 + 9 \pm \sqrt{(b^2+9)^2 + 4be^2}\right]}{2e}$

In order for c to be an integer, which it must be, the discriminant must be a perfect square. Test the eight possible solutions for (b,e):

For Comb. 2: $(b,e) \in \{(9,-3), (3,-9)\}$

For Comb. 3: $(b,e) = (3,-9)$

Check $(b,e) = (3,-9)$ in the full quadratics from Comb. 2 and Comb. 3. It checks.

If $(b,e) = (3,-9)$, then $c \in \{0, -3\}$ (the answers to the quadratics), but looking at equation 3 when solved for d ($d = (bc+e)/c$), we see that $c \neq 0$ since division by 0 is undefined. Therefore $c = -3$ and $a = -c = 3$.

It is now simple to solve for d; simply substitute the values of a, b, c, and e into any of the original equations containing d to yield $d = 6$.

Solution: $(a, b, c, d, e) = (3, 3, -3, 6, -9)$

The desired factorization is

$$(x^2 + 3x + 3)(x^3 - 3x^2 + 6x - 9) = x^5 - 9x - 27.$$

Solution 3 by Sarah Emerson (12/WA): Expand the equation to obtain:

$$x^5 + (a+c)x^4 + (ac+b+d)x^3 + (ad+bc+e)x^2 + (ae+bd)x + be = x^5 - 9x - 27$$

Therefore

(1) $a + c = 0$	$\Rightarrow$ $c = -a$
(2a) $d + ac + b = 0$	$\Rightarrow$ $d - a^2 + b = 0$
(3a) $e + ad + bc = 0$	$\Rightarrow$ $e + ad - ba = 0$
(4) $ae + bd = -9$	
(5) $be = -27$	

To solve, try all possible values of b and e, plugging the values into the other equations to determine if they work.

$b = 1$ $e = -27$	(4) $-27a + d = -9$ (3) $-27 + ad - a = 0$	$d = -9 - 27a$ $-27 + a(-9 - 27a) - a = 0$ $-27 - 9a - 27a^2 - a = 0$ $-27a^2 - 10a - 27 = 0$ $a = \dfrac{10 \pm \sqrt{100 - 4(-27)(-27)}}{2(-27)}$	a does not exist.
$b = 3$ $e = -9$	(4) $-9a + 3d = -9$ (3) $-9 + ad - 3a = 0$	$d = -3 + 3a$ $-9 - 3a + 3a^2 - 3a = 0$ $a^2 - 2a - 3 = 0$ $(a-3)(a+1) = 0$ $a = 3$ or $a = -1$	$a = 3$ $b = 3$ $c = -3$ $d = 6$ $e = -9$ ***it works! Solution.***

Editor's Comment: We thank our Problem Editor, Dr. George Berzsenyi, for this problem. It stems from a recent article, "The Factorization of $x^5 \pm p^2 x - k$ and Fibonacci Numbers," published in the November 1999 issue of the *Fibonacci Quarterly*.

4/1/12. A sequence of real numbers $s_0, s_1, s_2, \ldots$ has the property that $s_i s_j = s_{i+j} + s_{i-j}$ for all nonnegative integers i and j with $i \geq j$, $s_i = s_{i+12}$ for all nonnegative integers i, and $s_0 > s_1 > s_2 > 0$. Find the three numbers s_0, s_1, and s_2.

Solution 1 by Jennifer Dawson (11/AK):

Answer: $s_0 = 2$, $s_1 = \sqrt{3}$, and $s_2 = 1$.

Solution:

$$s_1 \cdot s_0 = s_1 + s_1 = 2s_1$$
$$s_0 = 2$$

$$s_1 \cdot s_1 = s_2 + s_0$$
$$s_2 = s_1^2 - s_0 = s_1^2 - 2$$

$$s_2 \cdot s_1 = s_3 + s_1$$

$$[s_1^2 - 2] \cdot s_1 = s_3 + s_1$$
$$s_3 = s_1^3 - 3s_1$$

$$s_3 \cdot s_3 = s_6 + s_0$$
$$[s_1^3 - 3s_1]^2 = s_6 + 2$$
$$s_1^6 - 6s_1^4 + 9s_1^2 = s_6 + 2$$
$$s_6 = s_1^6 - 6s_1^4 + 9s_1^2 - 2$$

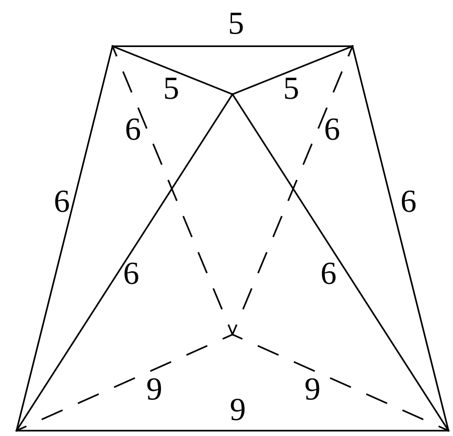
$$s_6 \cdot s_6 = s_{12} + s_0 = s_0 + s_0 = 4$$
$$[s_1^6 - 6s_1^4 + 9s_1^2 - 2]^2 = 4$$
$$s_1 \in \{-\sqrt{3}, \sqrt{3}, -2, 2, -1, 1, 0\}$$

Since s_1 must be strictly between 0 and 2, all but 1 and $\sqrt{3}$ are eliminated.
First, try $s_1 = 1$. $s_1 \cdot s_1 = s_2 + s_0$ yields $s_2 = -1$, a contradiction. So $s_1 \neq 1$.
Second, try $s_1 = \sqrt{3}$. $s_1 \cdot s_1 = s_2 + s_0$ yields $3 = s_2 + 2$ or $s_2 = 1$. It works!

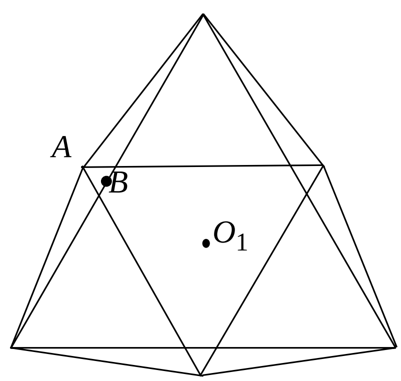
So, $s_0 = 2$, $s_1 = \sqrt{3}$, and $s_2 = 1$.

Editor's Comment: Once again, we are most grateful to Dr. Erin Schram of NSA for this intriguing problem. An indirect, but interesting solution begins with the observation that $\cos\alpha\cos\beta = \frac{1}{2}\cos(\alpha+\beta) + \frac{1}{2}\cos(\alpha-\beta)$ and modeling the sequence as $s_i = 2\cos(i\theta)$ for some θ.

5/1/12. In the octahedron shown on the right, the base and top faces are equilateral triangles with sides measuring 9 and 5 units, and the lateral edges are all of length 6 units. Determine the height of the octahedron; i.e., the distance between the base and the top face.



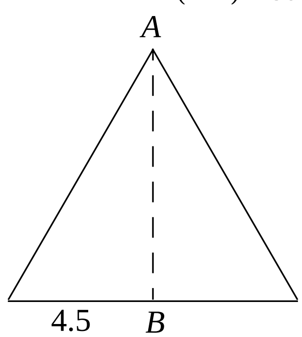
Solution 1 by Anna Maltseva (12/MI):
By symmetry, the top face is parallel to the base, and the line connecting the centers of the triangles of the top face and the base is perpendicular to both the top face and the base.

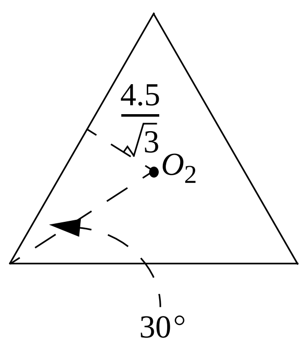


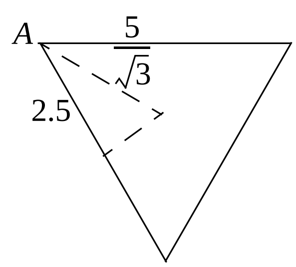
The top view of the octahedron looks like the figure at left. Let O_1 denote the center of the top face, O_2 denote the center of the base, A denote a vertex of the top face, and B denote the midpoint of the corresponding side of the base. Imagine dropping a line from A perpendicular to the base and let K denote the point where it intersects the plane of the base. Then triangle ABK will be a right triangle and AK will be the height of the octahedron.

So $\overline{AB}^2 = \overline{AK}^2 + \overline{BK}^2 = \overline{AK}^2 + (\overline{O_2B} - \overline{O_1A})^2$

$6^2 = \overline{AB}^2 + (4.5)^2$ so $\overline{AB}^2 = 15.75$.







Therefore,

$$15.75 = \overline{AK}^2 + \left(\frac{4.5}{\sqrt{3}} - \frac{5}{\sqrt{3}}\right)^2.$$

$$\frac{63}{4} - \frac{1}{12} = \overline{AK}^2$$

$$\overline{AK} = \sqrt{\frac{188}{12}} = \sqrt{\frac{47}{3}} = \frac{\sqrt{141}}{3}.$$

Solution 2 by Alexander Power (11/IA):
Let the equilateral triangle with side 5 have vertices *A*, *B*, and *C*, and let the equilateral triangle with side 9 have vertices *D*, *E*, and *F*, with sides of the octahedron *AD*, *BD*, *BE*, *CE*, *CF*, *AF*. Then, the height of the octahedron is the same as the height of tetrahedron *ABCD* and the same as the height of tetrahedron *CDEF*. We know the length of all the sides except *CD*. A formula for the volume of a tetrahedron *ABCD* is $\sqrt{\frac{T}{288}}$, where T is the determinant of the 5×5 matrix

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & AB^2 & AC^2 & AD^2 \\ 1 & AB^2 & 0 & BC^2 & BD^2 \\ 1 & AC^2 & BC^2 & 0 & CD^2 \\ 1 & AD^2 & BD^2 & CD^2 & 0 \end{bmatrix}$$

Plugging in values of the two tetrahedra, we get

$$T_1 = \det\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 25 & 25 & 36 \\ 1 & 25 & 0 & 25 & 36 \\ 1 & 25 & 25 & 0 & x^2 \\ 1 & 36 & 36 & x^2 & 0 \end{bmatrix} = -50x^4 + 4850x^2 - 6050$$

and

$$T_2 = \det\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 81 & 81 & 36 \\ 1 & 81 & 0 & 81 & 36 \\ 1 & 81 & 81 & 0 & x^2 \\ 1 & 36 & 36 & x^2 & 0 \end{bmatrix} = -162x^4 + 24786x^2 - 328050$$

as the determinants of the two tetrahedra, with T_1 as the determinant of tetrahedron *ABCD* and T_2 the determinant of *CDEF*. Since the tetrahedra have the same height, their volumes are proportional to the areas of their bases (by Cavalieri's Principle). The areas of their bases are proportional to a square of a side. Thus

$$81\sqrt{\frac{T_1}{288}} = 25\sqrt{\frac{T_2}{288}}.$$

This means that $T_2\left(\frac{25}{81}\right)^2 = T_1$. Thus

$$(-50x^4 + 4850x^2 - 6050) = (-162x^4 + 24786x^2 - 328050)\left(\frac{25}{81}\right)$$

or

$$\frac{-2800}{81}x^4 + \frac{22400}{9}x^2 + 25200 = 0$$

and x^2 is 81 or -9, but -9 is extraneous. This means that $CD = 9$. Thus, T_2 is

$$T_2 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 81 & 81 & 36 \\ 1 & 81 & 0 & 81 & 36 \\ 1 & 81 & 81 & 0 & 81 \\ 1 & 36 & 36 & 81 & 0 \end{bmatrix} = 616734$$

and the volume of tetrahedron *CDEF* is $\frac{27}{4}\sqrt{47}$. Since the base has area $\frac{81\sqrt{3}}{4}$, the height is

$$\sqrt{\frac{47}{3}} \sim 3.958 \ .$$

Editor's Comment: Dr. Berzsenyi based this problem on a recent article in *Mathematics Magazine* (vol. 72, no. 4, pp.277-286).