U S A Mathematical Talent Search

PROBLEMS / SOLUTIONS / COMMENTS
Round 4 - Year 11 - Academic Year 1999-2000

Gene A. Berg, Editor

1/4/11. Determine the unique 9-digit integer M that has the following properties: (1) its digits are all distinct and nonzero; and (2) for every positive integer $m = 2, 3, 4, ..., 9$, the integer formed by the leftmost m digits of M is divisible by m.

Solution 1 by Lisa Leung (9/MD): Let the 9-digit integer M be represented by *abcdefghi* where each letter represents a unique digit from the set of {1, 2, ...9}. Since *abcde* must be divisible by 5, e can only be 5 or 0. However the digits are non-zero, so $e = 5$. b, d, f, and h are even numbers since they are the last digits of numbers that are divisible by 2, 4, 6, and 8. When examining combinations of *cd* and *fgh*, where *cd* will be divisible by 4 and *fgh* will be divisible by 8, one can conclude from Table 1 (at right) that d is 2 or 6, and h is 2 or 6, since c and g cannot be even. The possible values for *cd* are 12, 16, 32, 36, 52, 56, 72, 76, 92, and 96.

c	***d***
1	2
1	6
2	4
2	8
3	2
3	6
4	8
5	2
5	6
6	4
6	8
7	2
7	6
8	4
9	2
9	6

Table 1:

The possible values for *fgh* are 416, 432, 472, 496, 816, 832, 872, 896. Table 2 (below) shows the possible digits of each position after the initial analysis. $a+b+c$ must be divisible by 3 and $d+e+f$ must be divisible by 3. The only possibilities of a, b, c, d, e, and f left are given in Table 3.

a	***b***	***c***	***d***	***e***	***f***	***g***	***h***	***i***
1		1	2	5		1	2	1
3	4	3			4	3		3
7		7	6			7	6	7
9	8	9			8	9		9

Table 2:

a	***b***	***c***	***d***	***e***	***f***	***ghi***
1	4	7	2	5	8	963
7	4	1	2	5	8	963
1	8	3	6	5	4	729
3	8	1	6	5	4	729
9	8	1	6	5	4	729

Table 3:

Among the five possibilities, only 3816547 is divisible by 7.

Thus, M = 381654729.

Solution 2 by Zhihao Liu (10/IL): Answer: 381654729.
Let *abcdefghi* $=M$, where letters a through i represent digits 1 through 9, let *abcde* represent a five digit number, and so on. Since $5 \mid$ *abcde* (read as, "5 divides *abcde*"), we conclude $e = 5$. Note that b, d, f, and h are all even, thus a, c, g, and i are odd. Since *abcd* and *abcdefgh* are divisible by 4 and their ten's digits are odd, we conclude that d and h are 2

and 6 (in some order), and b and f are 4 and 8 (in some order). Also, $3 \mid a+b+c$, $3 \mid a+b+c+d+e+f$, and $3 \mid a+b+c+d+e+f+g+h+i = 45$, so $3 \mid d+e+f$, and $3 \mid g+h+i$. From this we see that def is either 258 or 654. Since $8 \mid fgh$ and f is even, we conclude that $8 \mid gh$. With these restrictions, we eliminated all 9-digit numbers except: 147258963, 183654729, 189654327, 381654729, 741258963, 789654321, and 987654321. However, only 381654729 possesses the property that the number formed by its first seven digits is divisible by 7.

Therefore, ***M* = 381654729**.

Editor's Comment: We express thanks for this problem to Sándor Róka, the founding editor of *Abacus*, Hungary's mathematical journal for students age 10 - 14.

2/4/11. The Fibonacci numbers are defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n > 2$. It is well-known that the sum of any 10 consecutive Fibonacci numbers is divisible by 11. Determine the smallest integer N so that the sum of any N consecutive Fibonacci numbers is divisible by 12.

Solution 1 by Chi Cao Minh (12/TX): Let x and y represent two consecutive Fibonacci numbers. Continuing the sequence:

	Fibonacci Number	Sum of Sequence		Fibonacci Number	Sum of Sequence
1	x	x	13	$89x + 144y$	$233x + 376y$
2	y	$x + y$	14	$144x + 233y$	$377x + 609y$
3	$x + y$	$2x + 2y$	15	$233x + 377y$	$610x + 986y$
4	$x + 2y$	$3x + 4y$	16	$377x + 610y$	$987x + 1596y$
5	$2x + 3y$	$5x + 7y$	17	$610x + 987y$	$1597x + 2583y$
6	$3x + 5y$	$8x + 12y$	18	$987x + 1597y$	$2584x + 4180y$
7	$5x + 8y$	$13x + 20y$	19	$1597x + 2584y$	$4181x + 6764y$
8	$8x + 13y$	$21x + 33y$	20	$2584x + 4181y$	$6765x + 10945y$
9	$13x + 21y$	$34x + 54y$	21	$4181x + 6765y$	$10946x + 17710y$
10	$21x + 34y$	$55x + 88y$	22	$6765x + 10946y$	$17711x + 28656y$
11	$34x + 55y$	$89x + 143y$	23	$10946x + 17711y$	$28657x + 46367y$
12	$55x + 89y$	$144x + 232y$	24	$17711x + 28657y$	$46368x + 75024y$

Dividing the "Sum of the Sequence" by 12 shows that **the smallest number *N* is 24**.

Solution 2 by Ricky Liu (10/MA):
Suppose we have found N such that $F_k + F_{k+1} + \ldots + F_{k+N-1}$ is always divisible by 12. Then $(F_{k+1} + F_{k+2} + \ldots + F_{k+N}) - (F_k + F_{k+1} + \ldots + F_{k+N-1}) = F_{k+N} - F_k$ must be divisible by 12. In other words, given any two Fibonacci numbers with indices greater than one and separated by N, their difference must be divisible by 12. To simplify things, notice that if $F_2 \equiv F_{N+2} \pmod{12}$ and if $F_3 \equiv F_{N+3} \pmod{12}$, then $F_k \equiv F_{N+k} \pmod{12}$ for all $k > 1$. This is easily shown by induction: $F_{k+2} \equiv F_k + F_{k+1} \equiv F_{N+k} + F_{N+k+1} \equiv F_{N+k+2} \pmod{12}$. So, we need only find the second pair of consecutive Fibonacci numbers equivalent to 1 and 2 (mod 12) respectively. These occur at F_{26} and F_{27}. Thus, $N = 26 - 2 = 24$. $\boldsymbol{N = 24}$**.**

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Editor's Comment: This problem was created by the editor.

3/4/11. Determine the value of

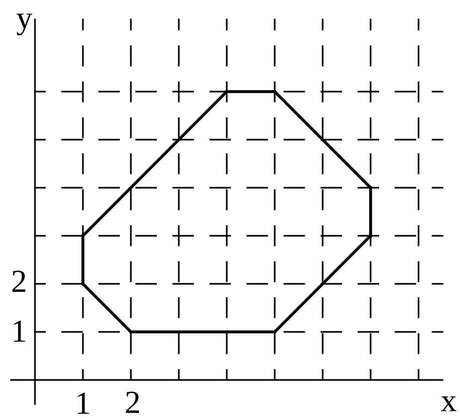
$$S = \sqrt{1 + \frac{1}{1^2} + \frac{1}{2^2}} + \sqrt{1 + \frac{1}{2^2} + \frac{1}{3^2}} + \ldots + \sqrt{1 + \frac{1}{1999^2} + \frac{1}{2000^2}}$$

Solution by Emily Kendall (11/IN):

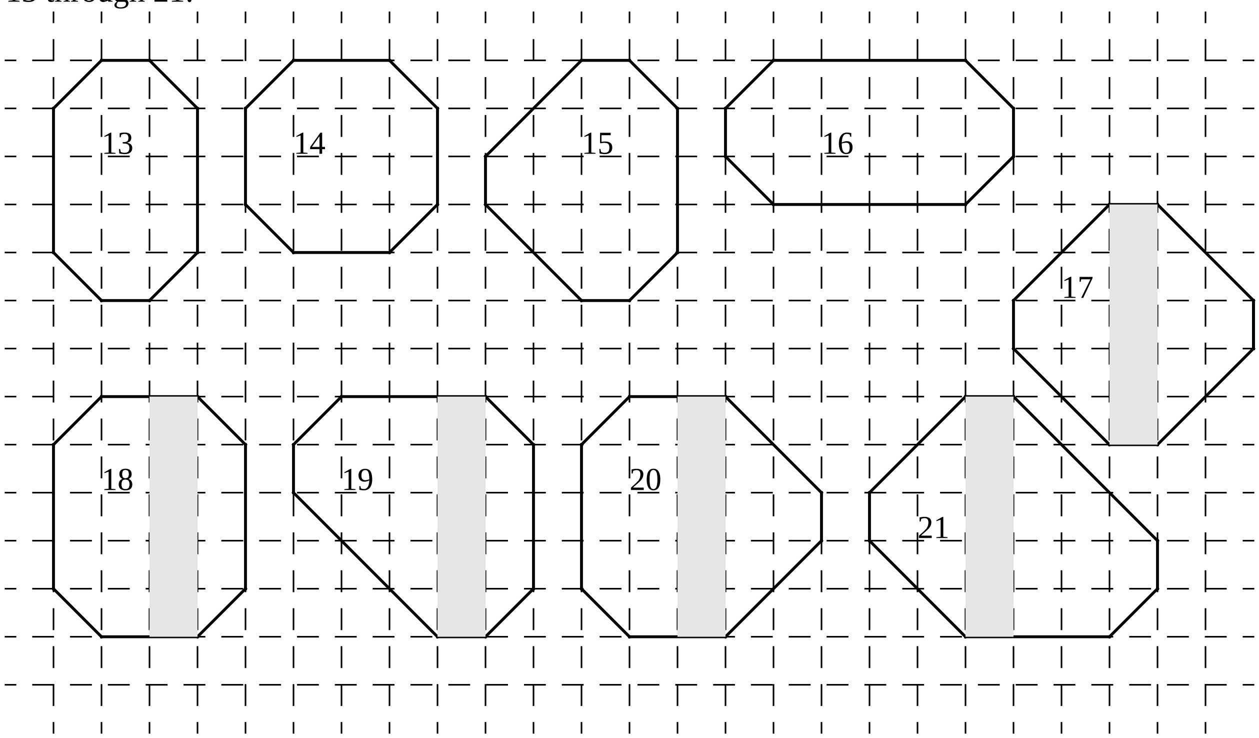
$$\begin{aligned}
S &= \sum_{a=1}^{1999} \sqrt{1 + \frac{1}{a^2} + \frac{1}{(a+1)^2}} \\
&= \sum_{a=1}^{1999} \sqrt{\frac{a^4 + 2a^3 + 3a^2 + 2a + 1}{a^2(a+1)^2}} \\
&= \sum_{a=1}^{1999} \frac{a^2 + a + 1}{a^2 + a} \\
&= \sum_{a=1}^{1999} \left(1 + \frac{1}{a^2 + a}\right) \\
&= 1999 + \sum_{a=1}^{1999} \left(\frac{1}{a} - \frac{1}{a+1}\right), \text{ a telescoping sum.} \\
&= 1999 + 1 - \frac{1}{2000} \\
&= 1999 + \frac{1999}{2000}
\end{aligned}$$

Editor's Comment: This timely problem was adapted by our Problem Editor, Dr. George Berzsenyi, from *MatLap*, Transylvania's Hungarian language mathematics journal for students at the middle and high school levels.

4/4/11. We will say that an octagon is integral if it is equiangular, its vertices are lattice points (i.e., points with integer coordinates), and its area is an integer. For example, the figure on the right shows an integral octagon of area 21. Determine, with proof, the smallest positive integer K so that for every positive integer $k \geq K$, there is an integral octagon of area k.



Solution 1 by Sawyer Tabony (11/VA): The integer K that I came up with is $K = 13$, and I will show you how I came up with this. First, I will show you that there is an integral octagon of area k for all k greater than or equal 13. Here are the diagrams of integral octagons of areas 13 through 21:



Notice that for the last five (17-21) there is a column shaded within each octagon exactly five units tall and one wide. To find an octagon of area 22, one would only need to widen the column in the octagon of area 17 by one unit. This would create the extra five square units. Since we have five consecutive octagons with this feature, it is clear that by expanding the width of these columns we can eventually reach any area.

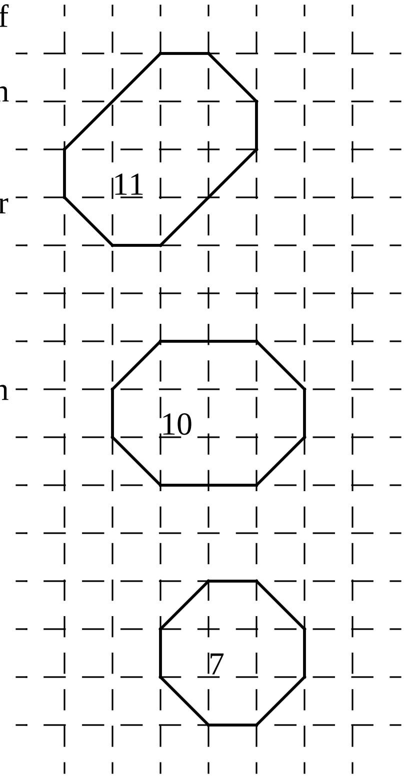
So now I must show that there does not exist an integral octagon of area 12. The smallest octagon whose height is five is the one of area 13 on the previous page, so clearly both height and width of an octagon of area 12 must be less than 5 (height and width are the dimensions of the smallest rectangle with vertical and horizontral sides into which the integral octagon fits). The largest four by four integral octagon is shown above and has an area of 14. The only other one is to the right, and it has area 11.

Next we move down to four by three octagons. There is only one octagon with dimensions four by three; it is shown at right (with an area of 10).

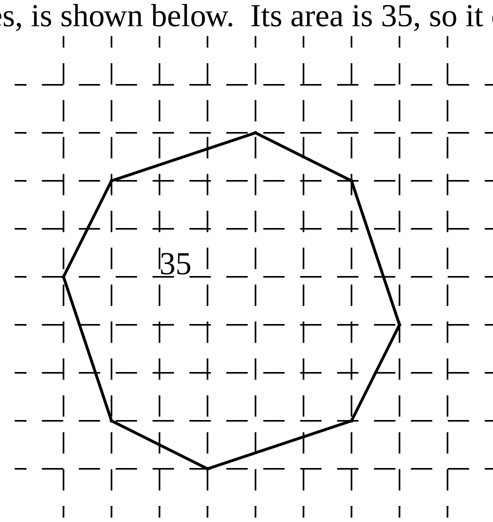
The final octagon is one of height and width three that has area of seven. This is shown at the right.

So there is no integral octagon of area 12.

This makes $\boldsymbol{K = 13}$.



Comment by Abhijit Mehta (10/OH): The smallest integral octagon, none of whose sides are parallel to the coordinate axes, is shown below. Its area is 35, so it does not alter the solution.

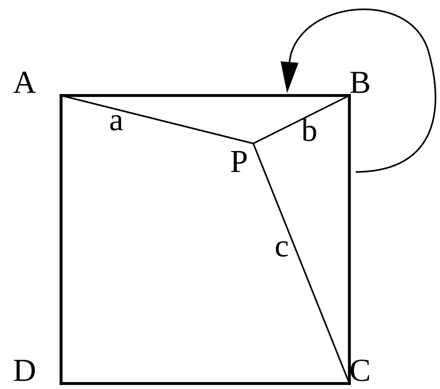


Editor's Comment: Dr. Berzsenyi based this problem on a similar, but more difficult problem, proposed by A. C. Heath in the March 1976 issue of *The Mathematical Gazette*.

5/4/11. (Revised 2-4-2000) Let P be a point interior to square $ABCD$ so that $PA = a$, $PB = b$, $PC = c$, and $c^2 = a^2 + 2b^2$. Given only the lengths a, b, and c, and using only a compass and straightedge, construct a square congruent to square ABCD.

Solution 1 by Nyssa Thompson (10/HI): I am assuming basic construction knowledge: constructing a right angle and a square with a given side length.

Connecting point *P* to points *A*, *B*, and *C* creates two triangles, ΔPAB and ΔPBC, that share the same side *PB*. Since *ABCD* is a square, *AB* = *BC* = *CD* = *DA*. We can refer to any of the sides as *s*. Furthermore, we can see that $\angle ABP + \angle CBP = 90°$ because they sum to $\angle ABC$ which is a corner of the square *ABCD*. Also, since it was given that $AP = a$, $BP = b$, and $CP = c$, we can refer to these lengths as such. Thus ΔPAB has sides of lengths *a*, *s*, and *b*, and ΔPBC has sides of lengths *b*, *s*, and *c*.



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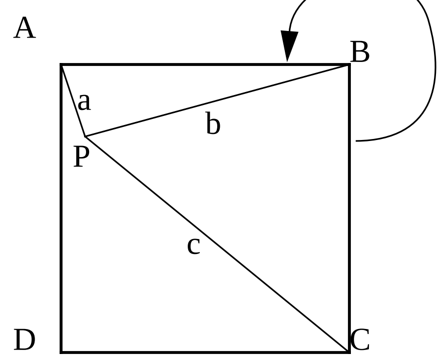
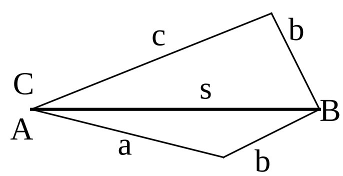


We can rotate these triangles so that instead of sharing side *b*, they share the *s* side. Then instead of the sides of length *s* forming a right angle, the sides of length *b* form a right angle (see below).



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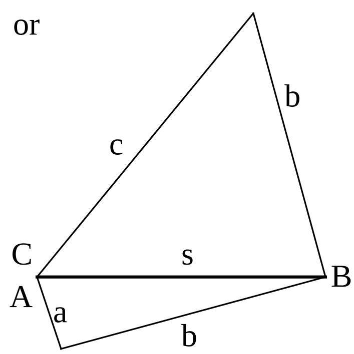

Thus, we have a polygon with two adjacent sides of length *b* forming a right angle at *B*, and remaining sides *a* and *c*. The segment from the vertex at *B* to the vertex at *A* and *C* is of length *s*.

With this knowledge we can construct the square *ABCD* if we are given *a*, *b*, and *c* using only compass and straight edge.

1. Construct a right angle.
2. Set the compass to length *b*, place its point on the vertex of the right angle, and mark off length *b* on each side of the angle.
3. Next set the compass to length *a* or *c*, place the compass point on the end of one of the lengths

b (not the vertex), and make a circle with the radius a or c.

4. Repeat step 3 with the remaining segment length a or c accordingly. Construct the circle on the opposite b's endpoint.
5. Now, these two circles should either intersect at two points or be tangent.
6. Choose the point which lies within the right angle and, using the straightedge, connect it to the vertex of the right angle. This length is s.
7. Construct a square with side length s. It will be congruent to square $ABCD$!

Solution 2 by Megan Guichard (12/WA): Let s be the length of the sides of $ABCD$ and let $\angle ABP = \theta$. Then, by the law of cosines, we have $a^2 = s^2 + b^2 - 2sb\cos(\theta)$, $\cos(\theta) = \frac{s^2+b^2-a^2}{2sb}$, and $\theta = \cos^{-1}\left(\frac{s^2+b^2-a^2}{2sb}\right)$. We also know, again by the law of cosines, that $c^2 = s^2 + b^2 - 2sb\cos(\angle CBP) = s^2 + b^2 - 2sb\cos(90° - \theta) = s^2 + b^2 - 2sb\sin(\theta)$. Since we know that $\theta = \cos^{-1}\left(\frac{s^2+b^2-a^2}{2sb}\right)$,

$$\begin{aligned}\sin(\theta) &= \sqrt{1-\cos^2(\theta)}\\ &= \sqrt{\frac{4s^2b^2 - s^4 + 2a^2s^2 - a^4 - 2s^2b^2 + 2a^2b^2 - b^4}{4s^2b^2}}\\ &= \frac{\sqrt{2a^2s^2 + 2s^2b^2 + 2a^2b^2 - s^4 - a^4 - b^4}}{2sb}\end{aligned}$$

Then, $2sb\sin(\theta) = \sqrt{2a^2s^2 + 2s^2b^2 + 2a^2b^2 - s^4 - a^4 - b^4}$. Since $c^2 = s^2 + b^2 - 2sb\sin(\theta)$,

$$2sb\sin(\theta) = s^2 + b^2 - c^2 = s^2 + b^2 - a^2 - 2b^2 = s^2 - a^2 - b^2$$

Squaring both sides of this equation gives

$$2a^2s^2 + 2s^2b^2 + 2a^2b^2 - s^4 - a^4 - b^4 = s^4 - 2a^2s^2 + a^4 - 2b^2s^2 + 2a^2b^2 + b^4$$

$$s^4 - 4a^2s^2 + 2a^4 - 4s^2b^2 + 2b^4 = 0$$

$$s^4 - 2(a^2+b^2)s^2 + (a^4+b^4) = 0$$

$$s^4 - 2(a^2+b^2)s^2 + (a^2+b^2)^2 = 2a^2b^2$$

$$(s^2 - (a^2+b^2))^2 = 2a^2b^2$$

$$s^2 - (a^2+b^2) = ab\sqrt{2}$$

$$s^2 = a^2 + b^2 + ab\sqrt{2}$$

$$s^2 = a^2 + b^2 - 2ab\left(-\frac{\sqrt{2}}{2}\right)$$
$$s^2 = a^2 + b^2 - 2ab\cos(135°).$$

Notice that this is simply the law of cosines for a triangle with sides *a*, *b*, and *s*, with the angle opposite *s* measuring $135°$. Since 2 sides and their included angle determine a triangle, we can construct *s* by constructing a triangle with two sides *a* and *b* such that the angle between *a* and *b* is $135°$. (We can construct an angle of $135°$ by constructing a $90°$ angle, bisecting it to get a $45°$ angle, and then adding another $90°$ angle.) Once we have *s*, we can construct a square congruent to *ABCD* fairly easily by constructing $90°$ angles and new sides of length *s* until we have a square with four sides of length *s*.

Editor's Comment: Notice that the first solution does not use $c^2 = a^2 + 2b^2$. Given that the square *ABCD* exists, does this construction work for any point P (interior or exterior) and set of distances *PA* = *a*, *PB* = *b*, *PC* = *c*? We are indebted to Professor Gregory Galperin of Eastern Illinois University for suggesting this wonderful problem.