U S A Mathematical Talent Search

PROBLEMS / SOLUTIONS / COMMENTS
Round 2 - Year 11 - Academic Year 1999-2000

Gene A. Berg, Editor

1/2/11. The number N consists of 1999 digits such that if each pair of consecutive digits in N were viewed as a two-digit number, then that number would either be a multiple of 17 or a multiple of 23. The sum of the digits of N is 9599. Determine the rightmost ten digits of N.

Solution 1 by Ann Chi (8/IN):
Multiples of 17: 17, 34, 51, 68, 85. Multiples of 23: 23, 46, 69, 92.

68,85,51,17 can be used only at the very end because no other numbers start in 8, 5, 1, or 7.
Another possible sequence of numbers is 23,34,46,69,92,23,... starting with any of these numbers.

$$2 + 3 + 4 + 6 + 9 = 24$$
$$1999 \div 5 = 399\,\mathrm{R}4$$
$$399 \times 24 = 9576$$
$$9599 - 9576 = 23$$

So there must be an end sequence of 4 numbers adding up to 23. Since picking any four of the original repeating sequence will not add up to 23, we have to use part of the sequence 923468517 (there must be a 46 before the 68).
4+6+8+5 = 23. This means the sequence of numbers must be

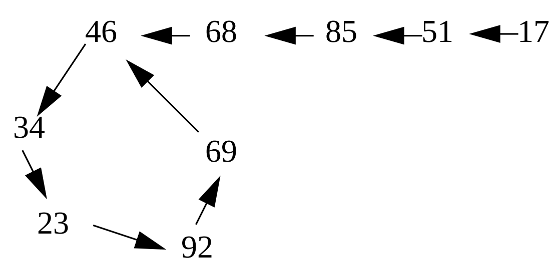
469234692346...4685.

The last ten digits of this sequence are **3469234685**.

Solution 2 by Nick Masiewicki (11/MD): Consider all two-digit multiples of 17 and 23:

17, 23, 34, 46, 51, 68, 69, 85, 92.

Each digit 1-9 is represented exactly once as a 1's digit of a number on that list. Thus, one and only one of those numbers may precede a number on that list in *N* in the sense that *ab* precedes *bc* in *abc*. Thus we know 51 precedes 17, 85 precedes 51, and so on (see diagram at right). Thus, no matter what the rightmost digit of *N* is, we'll always get to the sequence 69234 (read left to right) repeated for the rest of *N*. Since that sequence is at most four steps away from any digit, that sequence must repeat $\frac{1999-4}{5} = 399$ times. The sum of its digits is 24. Thus we have accounted for $399 \times 24 = 9576$ of the sum. That leaves 23 left. We must find a four-member subset of consecutive digits in 346923468517 whose sum is 23. 4685 is the only such subset. Thus the rightmost ten digits of *N* are **3469234685**.



Editor's Comment: Once again, we are indebted to Professor Sándor Róka, the founding editor of Hungary's *Abacus*, for the idea of this problem. For an English translation of the problems and for their solution, the reader is referred to: http://www.gcschool.org/abacus.html. This competition is open to all students in grades 3 to 8.

2/2/11. Let C be the set of non-negative integers which can be expressed as $1999s + 2000t$, where s and t are also non-negative integers.

(a) Show that 3,994,001 is not in C.

(b) Show that if $0 \le n \le 3{,}994{,}001$ and n is an integer not in C, then $3{,}994{,}001 - n$ is in C.

Solution by David Gaebler (12/IA):

(a) Proof by contradiction: Suppose instead that 3,994,001 is in C, so $1999s + 2000t = 3994001$. Taking both sides mod 1999, $t \equiv 1998 (\mathrm{mod}\, 1999)$. Since $t \ge 0$, this implies $t \ge 1998$, so $2000t \ge 3996000$. But then s must be negative, contrary to the hypothesis that 3994001 is in C. Thus, 3,994,001 is not in C.

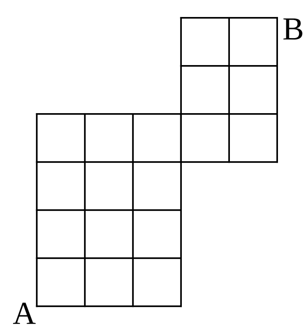
(b) Let $n = 1999a + b$ where $0 \le b \le 1998$. Since $n = 1999(a - b) + 2000b$, if $n \notin C$, then $b > a$.

$$3994001 - n = (2000 \cdot 1998 - 1999) - (1999a + b)$$
$$2000(1998 - b) + 1999(b - a - 1)$$

Since $b > a$, $b - a - 1 \ge 0$. Since $b \le 1998$, $1998 - b \ge 0$.
Therefore, 3,994,001 - n is in C.

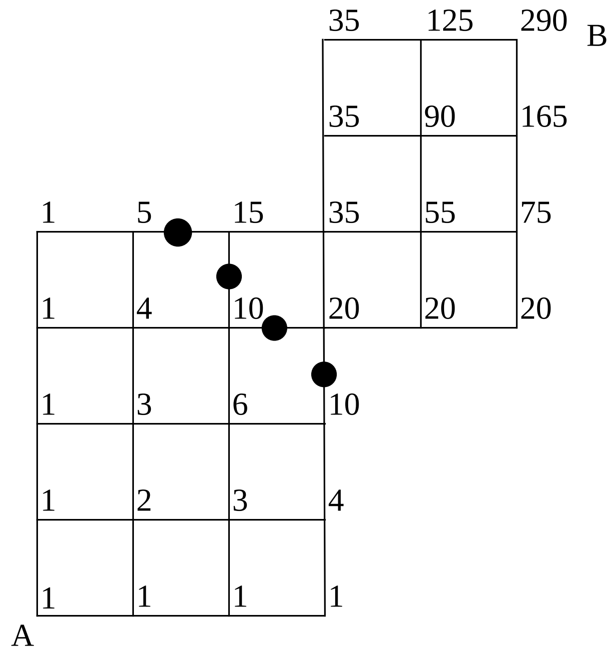
Editor's Comment: Appreciation is hereby expressed to Dr. Peter Anspach of the National Security Agency for proposing this problem.

3/2/11. The figure on the right shows the map of Squareville, where each city block is of the same length. Two friends, Alexandra and Brianna, live at the corners marked by A and B, respectively. They start walking toward each other's house, leaving at the same time, walking with the same speed, and independently choosing a path to the other's house with uniform distribution out of all possible minimum-distance paths [that is, all minimum-distance paths are equally likely]. What is the probability they will meet?



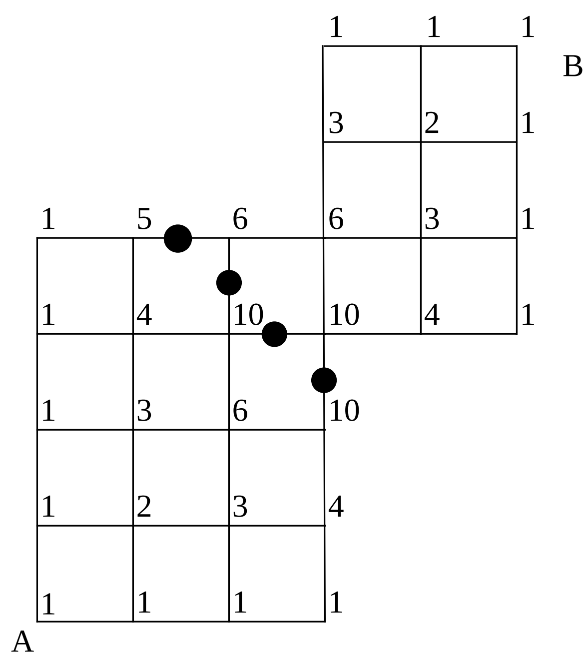
Solution 1 by Seth Kleinerman (12/NY):

The number of paths from A to B is equal to the number of paths from B to A. To determine this number, we note that the number of paths from A to a vertex is equal to the sum of the number of paths from A to the vertex below it and the number of paths from A to the vertex to the left of it. So starting at A, we can inductively find the number of paths from A to B as shown on the left.



There are 290 possible paths from A to B. If the two girls are to meet somewhere in the middle, they must have each traveled 5 1/2 blocks, since the total trip is 11 blocks and they start at the same time and walk at the same speed. They can only meet in the middle of one of the four marked streets in the above diagram (not at a street corner), and if they cross the same marked street they must meet. We shall call the streets 1, 2, 3, and 4 from top-left to bottom-right.

Let us repeat the above inductive process to determine the number of paths from both A and B to the corners lying on either side of these streets (see diagram at right). There are $5 \cdot 6 = 30$ paths going through street 1, $6 \cdot 10 = 60$ paths going through street 2, $10 \cdot 10 = 100$ paths going through street 3, and $10 \cdot 10 = 100$ paths going through street 4. So the probability of crossing each of these streets is 30/290, 60/290, 100/290, and 100/290 for streets 1, 2, 3, and 4 respectively.



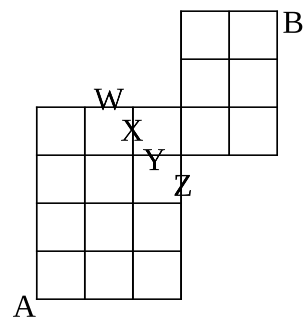
The probability that both girls will choose to cross at street 1 is $(30/290)^2$, that both will choose to cross at street 2 is $(60/290)^2$, and so on. So the probability that they meet (equivalently stated, the probability of both choosing the same numbered street to cross) is

$$\frac{30^2 + 60^2 + 100^2 + 100^2}{290^2} = \frac{245}{841}$$

Solution 2 by Andrew Dudzik (11/CA):

Lemma: The number of equal-distance paths from one corner of an a by b rectangle is $\binom{a+b}{a}$.

Proof of lemma: The total distance to be traveled is $a + b$. At any of these $a + b$ steps, we can choose to go down instead of left, but we must do so exactly a times (or b times, depending on the orientation). So the total number is $\binom{a+b}{a}$.



Problem solution: See the diagram at the right. Because Alexandra and Brianna are moving at the same speed and start at the same time, they can only meet at one of the streets labeled *W, X, Y,* and *Z*. The number of paths that go through any given street is equal to the number that go from *A* to one side of it times the number of paths that go from *B* to the other side of it. So the number of paths through *W* is $\binom{5}{1} \cdot \binom{4}{2} = 30$. The number through *X, Y,* and *Z* are $\binom{5}{2} \cdot \binom{4}{2} = 60$, $\binom{5}{2} \cdot \binom{5}{2} = 100$, and $\binom{5}{2} \cdot \binom{5}{2} = 100$, respectively. Since these choices are mutually exclusive, the probability that *A* will select the same street as *B* is

$$\left(\frac{30}{290}\right)^2 + \left(\frac{60}{290}\right)^2 + \left(\frac{100}{290}\right)^2 + \left(\frac{100}{290}\right)^2 = \frac{245}{841}$$

Solution 3 by Megan Guichard (12/WA):

Label the coordinates of the corners on the map as if they were points in the plane, using one city block as a unit, with the origin at *A*. Then A has coordinates (0,0) and B has coordinates (5,6). Since all minimum-length paths are 11 units long, Alexandra and Brianna will meet only if their paths contain the same block as the sixth block each passes over; that is, they must both choose the segment from (1,4) to (2,4), the segment from (2,3) to (2,4), the segment from (2,3) to (3,3), or the segment from (3,2) to (3,3). Since there are the same number of paths going from A to B as going from B to A, it is sufficient to find the probability of passing over a given one of these four crucial segments while traveling from A to B. If we start from (1,4) and go through (2,4), there is one way to get to the point (3,4), one way to get from there to point (3,5), and one way to get from (3,4) to point (4,4). After this there is one way to get to (3,6), two ways to get to (4,5), and one way to get to (5,4). Then there are three ways to get to each of (4,6) and (5,5), and finally, six ways to get to B from (1,4). Carrying out similar calculations for the other three crucial segments, there are 6 ways to get from B to (2,4), 10 ways to get from B to (3,3), 5 ways to get from A to (1,4), 10 ways to get from A to (2,3), and 10 ways to get from A to (3,2).

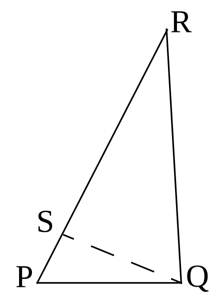
Therefore, there are a total of $5 \cdot 6 = 30$ ways to go from A to B passing through the segment from (1,4) to (2,4); there are a total of $10 \cdot 6 = 60$ ways to go from A to B passing through the segment from (2,3) to (2,4); there are $10 \cdot 10 = 100$ ways to go from A to B passing through the segment from (2,3) to (3,3); and $10 \cdot 10 = 100$ ways to go from A to B passing through the segment from (3,2) to (3,3). There are $30 + 60 + 100 + 100 = 290$ ways to get from A to B.

We then calculate the relevant probabilities. There is a $\frac{30^2}{290^2}$ chance that both Alexandra and Brianna will pass through the first crucial segment, a $\frac{60^2}{290^2}$ chance that both will pass through the second segment, a $\frac{100^2}{290^2}$ chance that both will pass through the third segment, and a $\frac{100^2}{290^2}$ chance that both will pass through the fourth segment.

Adding these probabilities together results in $\frac{245}{841}$, the chance that Alexandra and Brianna will meet.

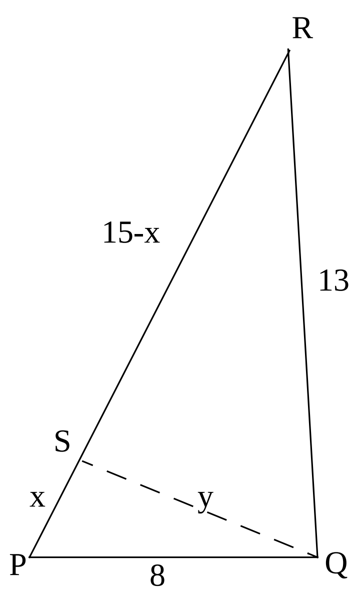
Editor's Comment: We thank our problem editor, Dr. George Berzsenyi, for this problem**.**

4/2/11. In ΔPQR, $PQ = 8$, $QR = 13$, and $RP = 15$. Prove that there is a point S on line segment $\overline{PR}$, but not at its endpoints, such that PS and QS are also integers.



Solution 1 by Frank Chemotti (11/WI): By the law of cosines:

$$13^2 = 15^2 + 8^2 - 2 \cdot 15 \cdot 8 \cdot \cos\angle RPQ$$
$$\cos\angle RPQ = 0.5$$



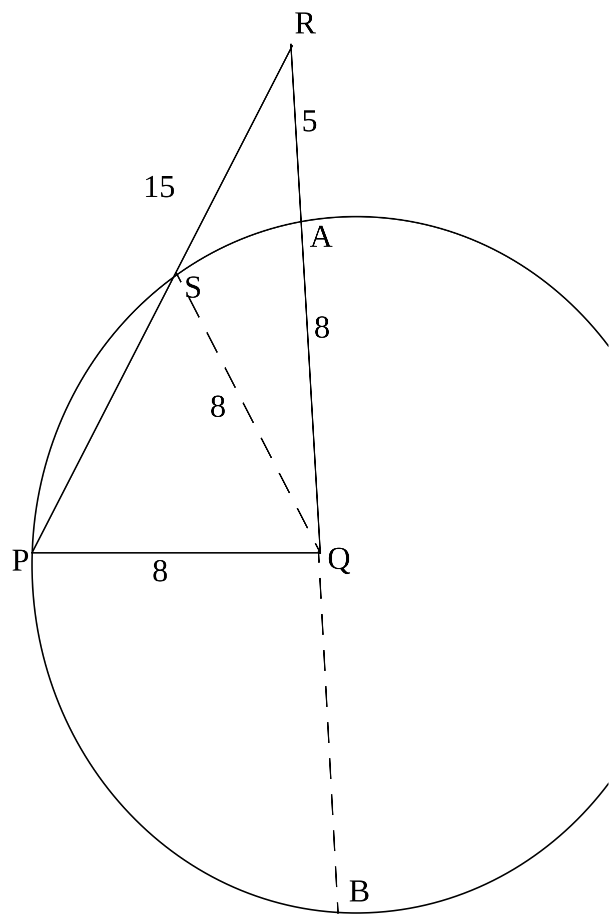
Let x be the length of segment PS and y be the length of segment QS. By the law of cosines:

$$y^2 = x^2 + 8^2 - 2 \cdot x \cdot 8 \cdot \cos\angle RPQ$$
$$y^2 = x^2 + 8^2 - 2 \cdot x \cdot 8 \cdot (0.5)$$
$$y^2 = x^2 - 8x + 64$$
$$y = \sqrt{x^2 - 8x + 64}$$
$$(0 < x < 15)$$

Integer solutions exist for this equation: (3, 7), (5, 7), and (8, 8).

Therefore, PS and SQ are both integers when point S is **3, 5, or 8** units away from point P.

Solution 2 by Daren Zou (12/NC): Draw circle Q with radius 8. Since $\angle RPQ$ is not opposite the longest side of the triangle, it must be acute. Intersect PR at S.



By Power of the Point Theorem

$$RA \cdot RB = RS \cdot RP$$
$$5 \cdot 21 = RS \cdot 15$$
$$RS = 7$$

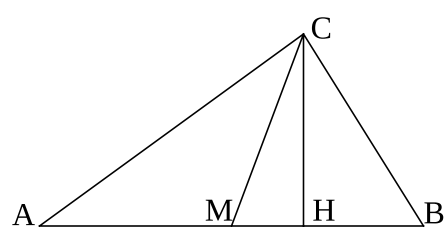
So $PS = 8 = QS$ are both integers.

Editor's Comment: Although this solution does not give all solutions, it is unique and quite interesting. To find the remaining two solutions, draw circle Q with radius 7 (rather than 8) and show RP intersects the circle in two points S and S'. Then, to show the distances from P to the two points are 3 and 5, apply the Power of the Point Theorem twice, using lines PQB' and RSS'.

Using this theorem, can you show these are all the points?

This problem was also created by Dr. George Berzsenyi.

5/2/11. In ΔABC, AC > BC, CM is the median, and CH is the altitude emanating from C, as shown in the figure on the right. Determine the measure of $\angle MCH$ if $\angle ACM$ and $\angle BCH$ each have measure $17°$.



Solution 1 by Yuran Lu (11/ME): Let us prove the general case:

Lemma. If $\angle ACM = \angle BCH = \alpha$ and $\angle MCH = \theta$, then $2\alpha + \theta = \frac{\pi}{2}$.

Proof of lemma. Let $CH = 1$. $AM = AH - MH$, $BM = BH + MH$.
$AH = 1 \cdot \tan(\alpha + \theta)$, $BH = 1 \cdot \tan\alpha$, and $MH = 1 \cdot \tan\theta$.
Since AM = BM we have:

$$1 \cdot \tan(\alpha + \theta) - 1 \cdot \tan\theta = 1 \cdot \tan\alpha + 1 \cdot \tan\theta$$

$$\frac{\tan\alpha + \tan\theta}{1 - \tan\alpha\tan\theta} - \tan\theta = \tan\alpha + \tan\theta$$

$$\tan\alpha + \tan\theta = (\tan\alpha + 2\tan\theta)(1 - \tan\alpha\tan\theta)$$

$$\tan\alpha + \tan\theta = \tan\alpha + 2\tan\theta - \tan^2\alpha\tan\theta - 2\tan\alpha\tan^2\theta$$

$$\tan\theta - \tan^2\alpha\tan\theta - 2\tan\alpha\tan^2\theta = 0$$

$$1 - \tan^2\alpha - 2\tan\alpha\tan\theta = 0$$

$$\tan\theta = \frac{1 - \tan^2\alpha}{2\tan\alpha} = \frac{1}{\tan 2\alpha} = \cot 2\alpha = \tan\left(\frac{\pi}{2} - 2\alpha\right)$$

$$\theta = \frac{\pi}{2} - 2\alpha$$

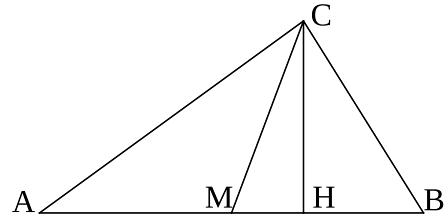
This completes the proof of the lemma.

Problem solution: If $\angle ACM = \angle BCH = 17$, then by the lemma

$$\angle MCH = 90° - 2(17°) = 56°.$$

Solution 2 by Jacqueline Ou (11/MA):

Let $\angle MCH = x°$. Then $\angle HBC = 73°$ and $\angle CAH = (73 - x)°$. We also know that $AM = BM$ since M is the midpoint of AB.



Applying the law of sines to ΔACM and to ΔBCM, we obtain

$$\frac{\sin 17}{AM} = \frac{\sin(73 - x)}{CM} \text{ and } \frac{\sin(17 + x)}{AM} = \frac{\sin(73)}{CM}$$

So

$$\frac{\sin 17}{\sin(17 + x)} = \frac{\sin(73 - x)}{\sin 73}$$

Expanding $\sin(17 + x)$ and $\sin(73 - x)$, and cross multiplying, we obtain

$$(\sin 73\cos 17 - \sin 17\cos 73)\cdot(\sin x\cos x) + \sin 17\sin 73\cos 2x = \sin 17\sin 73$$

$$\sin 56\sin x(\sqrt{1 - \sin^2 x}) + \sin 17\sin 73 - 2\sin 17\cos 17\sin^2 x = \sin 17\sin 73$$

$$\sin 56(\sqrt{1 - \sin^2 x}) - \sin 34\sin x = 0$$

$$\sin^2 56 - \sin^2 56\sin^2 x = \sin^2 34\sin^2 x$$

$$\sin^2 56 = \sin^2 x$$

$$x = 56°.$$

Editor's Comment: We are thankful to Professor Gregory Galperin for this problem and for his many other contributions to the USAMTS.