U S A Mathematical Talent Search

PROBLEMS / SOLUTIONS / COMMENTS
Round 1 - Year 11 - Academic Year 1999-2000

Gene A. Berg, Editor

1/1/11. The digits of the three-digit integers *a, b,* and *c* are the nine nonzero digits 1, 2, 3, ..., 9, each of them appearing exactly once. Given that the ratio *a:b:c* is 1:3:5, determine *a, b,* and *c*.

Solution 1 by Antoinette Realica (10/GU):

a:	*b:*	*c*
1:	3:	5
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a, b, c have 9 nonzero digits with each of them appearing exactly once.

The smallest possible number: 123

The last digit of *c* should be a multiple of 5 ending in 5, so the first digit of *a* should end in an odd digit (3, 7, or 9). The first digit of *a* should be 1 because if it were larger, then *c* would be four digits.

Process of trial and error:

a:	*b:*	*c*	
1:	3:	5	
12**3**:	**36**9	**6**15	3 and 6 repeated
127:	**3**8**1**:	6**3**5	1 and 3 repeated
129:	387	645	This works!

a=129, b = 387, c = 645.

Solution 2 by Vladimir Novakovski (10/VA): Let $a = a_2a_1a_0$, $b = b_2b_1b_0$, $c = c_2c_1c_0$.

Since $c = 5a < 1000$, $a < 200$, so $a_2 = 1$. Let us examine a_0. If a_0 is even, then $c_0 = 0$, which is impossible. If $a_0 = 5$, then $b_0 = c_0 = 5$, which is also impossible. If $a_0 = 7$, then $b_0 = 1$, which is impossible as well (since $a_2 = 1$). So $a_0 \in \{3, 9\}$.

Case 1: $a_0 = 3$

Consider a_1, b_1, and c_1. From our hypothesis, it follows that $b_1 \equiv 3a_1 \pmod{10}$, and that $c_1 \equiv 5a_1 + 1 \pmod{10}$ (the latter since $5 \cdot 3 = 10 + 5$, and the 10 contributes to the tenths digits). Therefore a_1 cannot be even, since then $c_1 = 1$. Also, $a_1 \neq 5$ since otherwise $a_1 = b_1 = 5$;

and $a_1 \neq 7$ since otherwise $b_1 = 1$. Since $a_0 = 3$, $a_1 \neq 3$, and since $b_0 \equiv 3 \cdot a_0 \bmod 10 = 9$, $a_1 \neq 9$. So there are no solutions.

Case 2: $a_0 = 9$

Again, consider a_1, b_1, and c_1. Now $b_1 \equiv 3a_1 + 2 \pmod{10}$, and $c_1 \equiv 5a_1 + 4 \pmod{10}$. Now $a_1 \neq 3$ since then $b_1 = 1$; $a_1 \neq 4$ since then $b_1 = 4$; $a_1 \neq 5$ since then $c_1 = 9$; $a_1 \neq 6$ since then $b_1 = 0$; $a_1 \neq 7$ since then $c_1 = 9$; $a_1 \neq 1$ and $a_1 \neq 9$. So the only possibilities are 2 and 8. Now $189 \cdot 5 > 900$, but $c_0 \neq 9$. It can be checked by arithmetic that 129 does work, so the numbers ***a***, ***b***, and ***c*** are **129**, **387**, and **645**.

Editor's Comments: This problem was proposed by Professor Sándor Róka, the founding editor of Hungary's *Abacus*, a problem solving journal for students in grades 4 through 8.

2/1/11. Let N = 111...1222...2, where there are 1999 digits of 1 followed by 1999 digits of 2. Express N as the product of four integers, each of them greater than 1.

Solution 1 by Kirsten Rutschman (12/WA):
To solve, look for patterns:

$1122 \div 11 = 102$

$111222 \div 111 = 1002$

$11112222 \div 1111 = 10002$

$N \div A = B$ where A is 1999 digits of 1 and B = 1000...02 has 1998 digits of 0.

$102 \div 2 = 51$

$1002 \div 2 = 501$

$10002 \div 2 = 5001$

$B \div 2 = C$ where C = 5000...01 with 1997 digits of 0.

$51 \div 3 = 17$

$501 \div 3 = 167$

$5001 \div 3 = 1667$

$C \div 3 = D$ where D = 1666...67 with 1997 digits of 6.

$N = A \cdot B \cdot C \cdot D = 111\ldots1 \cdot 2 \cdot 3 \cdot 1666\ldots67$.

Solution 2 by Luke Gustafson (11/MN):

$10^n - 1$ is a string of n 9's, hence $(10^n - 1)/9$ is a string of n ones. Then we have

$$N = 10^{1999}\left(\frac{10^{1999} - 1}{9}\right) + 2\left(\frac{10^{1999} - 1}{9}\right)$$

$$N = \left(\frac{10^{1999}-1}{9}\right)(10^{1999}+2)$$

$$N = 2\cdot\left(\frac{10^{1999}-1}{9}\right)(5\cdot 10^{1998}+1)$$

$$N = 2\cdot 3\cdot\left(\frac{10^{1999}-1}{9}\right)\left(\frac{5\cdot 10^{1998}+1}{3}\right)$$

The last factor is an integer since the sum of the digits of $5\cdot 10^{1998}+1$ (a five followed by a string of zeros and then a one) is 6, implying it is divisible by three. Hence

$$N = 2\cdot 3\cdot(111\ldots 1)\cdot(1666\ldots 67)$$

It might be interesting to note that N is also divisible by 23. I will show that $5\cdot 10^{1998}+1$ is divisible by 23. This proof makes use of Fermat's Theorem: if $x \neq 0 (\mathrm{mod} 23)$, then $x^{22} \equiv 1 (\mathrm{mod} 23)$.

$$\begin{aligned} 5\cdot 10^{1998}+1 &\equiv 5\cdot(10^{22})^{90}(10^{18})+1 (\mathrm{mod} 23) \\ &\equiv 5\cdot 10^{18}+1 \\ &\equiv 5\cdot(100^{9})+1 \\ &\equiv 5\cdot 8^{9}+1 \\ &\equiv 5\cdot(2^{22})\cdot(2^{5})+1 \\ &\equiv 5\cdot 2^{5}+1 \\ &\equiv 161 \\ &\equiv 0 (\mathrm{mod} 23) \end{aligned}$$

This yields a more complete factorization of *N*:

$$N = 2\cdot 3\cdot 23\cdot\left(\frac{10^{1999}-1}{9}\right)\cdot\left(\frac{5\cdot 10^{1998}+1}{69}\right)$$

Solution 3 by Alex Utter (11/CA): *N* is even so it is a multiple of 2, but the last two digits are not divisible by 4, so *N* is not a multiple of 4. The sum of its digits is $1999\cdot 1 + 1999\cdot 2 = 1999\cdot 3$, so it is a multiple of 3 but not 9. 5 is not a factor because *N* does not end with 0 or 5.

Potential third factors can be tested by a technique that I devised based on remainder repetition intervals. I will use the division of 7 into a short series of ones and twos as an example:

$$
\begin{array}{r|l}
 & \;\;1\quad 5\quad 8\quad 7\quad 3\quad 0\quad 3\quad 1\quad 7\quad 4\quad 6 \\
\hline
7 & 1_1\ 1_4\ 1_6\ 1_5\ 1_2\ 1_0\ 2_2\ 2_1\ 2_5\ 2_3\ 2_4\ 2_0
\end{array}
$$

As you can see, it divides evenly into 6 ones, and divides evenly into 6 twos. Because the remainder at the end of each section is zero, it can loop around as many times as desired. In addition, at equal distances from the change from 1 to 2 are two remainders of 2, so the cycle can be interrupted at that point. Therefore, 7 is a factor of any number like N with $6x$ ones and $6x$ twos, or a number with $6x+5$ ones and $6x+5$ twos, with x as an integer greater than or equal to 0. I write this cycle structure as $6x$ and $6x+5$. Since 1999 does not equal $6x$ or $6x+5$ for any integer x, 1999 does not fit seven's cycle, and 7 is not a factor.

Now, this method of testing primes until a factor is found can be used. The results starting after 7 are:

Prime	Cycle structure
11	*2x*
13	*6x*
17	*16x, 16x+2*
19	*18x, 18x+8*
23	*22x*, ***22x+19***

$1999 = 22 \cdot 90 + 19 = 22 \cdot x + 19$, so the third factor of N is 23. The results of division cycle with the remainder cycles, so N/23 can be expressed as follows:

(Notation: X indicates that the previous series of numbers should be repeated the indicated number of times, ... indicates that the next number should be concatenated to the previous one)

N/23 = 004830917874396135265 7X90...
004830917874396135266183574879227...
0096618357487922705314X90.

Each section alone is not a multiple of 3, but the three sections together are. In addition, the middle section (put together) is also a multiple of 3. So

$$\frac{N}{3 \cdot 23} =$$

0016103059581320450885668276972624798711755233494363929146537842 19X30...
0016130595813204508872785829307568438...
003220611916264090177133655394524959742351046698872785829307568438X30

Only the first section is odd, but the remainder can be contained in the first digit of the next repetition. Therefore, the final solution is

***N* = 2 * 3 * 23 ***
280008051529790660225544283413848631239935587761672718196457326892109...
500805152979066022544283413848631239935587761674718196457326892109X29...
500805152979066022544363929146537842 19...
0016103059581320454885668276972624798711755233494363929146537842 19X30

Editor's Comments: We thank Dr. George Berzsenyi, the creator of the USAMTS and now retired from Rose-Hulman Institute of Technology to the mountains of Colorado, for this beautiful problem.

The solutions to these problems give us an opportunity to briefly introduce congruences and modular arithmetic, and state an important theorem from Number Theory. Our goal is to introduce some of the notation and terminology, and briefly introduce this subject to young mathematicians seeing this for the first time, and possibly help them understand the proofs.

Two integers *a* and *b* are said to be *congruent modulo m* if *m* divides $a - b$. We write $a \equiv b(\text{mod}\, m)$, and say that *a* is congruent to *b* modulo *m*. The positive integer *m* is called the *modulus*, and the relation is called a *congruence*. For example,

$$86 \equiv 6(\text{mod}\,10) \qquad 1 \equiv 16(\text{mod}\,5) \qquad -3 \equiv 4(\text{mod}\,7) \qquad 88 \equiv 0(\text{mod}\,11)$$

Observe that if you divide *a* by *m* and get remainder *r*, then $a \equiv r(\text{mod}\, m)$. If $a \equiv b(\text{mod}\, m)$ and $c \equiv d(\text{mod}\, m)$, then $a + b \equiv c + d(\text{mod}\, m)$ and $a \cdot b \equiv c \cdot d(\text{mod}\, m)$.

This gives us an interesting new arithmetic system where we can add, subtract, and multiply. We refer to this arithmetic system as "arithmetic modulo *m*". [Do you see a connection between "clock arithmetic" and arithmetic modulo 12?]

If *m* is a prime integer, then we can also divide by nonzero elements. For example, observe that $2 \cdot 4 \equiv 1(\text{mod}\ 7)$, so we think of 2 as the multiplicative inverse of 4; that is, $2 \equiv \frac{1}{4}$ in this arithmetic system. Thus, dividing by 4 is the same as multiplying by 2 when you are working modulo 7. Whenever *m* is a prime number, and *a* is *not* congruent to 0 modulo *m*, then we can find a second number *b* such that $a \cdot b \equiv 1(\text{mod}\, m)$; dividing by *a* is the same as multiplying by *b*, so we now can divide by any "nonzero" element *a*. Now we have an arithmetic system where we can add, subtract, multiply, and divide by nonzero numbers.

This sounds good, you say, but given *a*, how do you find the corresponding *b*? The process is straightforward and efficient, and uses the extended Euclidean algorithm for finding Greatest Common Divisors, an algorithm already familiar to you, perhaps. If *p* is prime and *p* does not divide *a*, then GCD(*p*,*a*) = 1. Therefore there are integers *r* and *s* such that 1 = *ar* + *ps*. So $1 \equiv ar + ps(\text{mod}\, p) \equiv ar(\text{mod}\, p)$, and *r* is the multiplicative inverse of *a*, as desired.

As an example, we will find 19^{-1} modulo 61. That is, we find an integer r such that $19 \cdot r \equiv 1 \pmod{61}$. First we use the division algorithm, divide 61 by 19, and get remainder 4:

$$4 = 61 - (3 \cdot 19)$$

Next, divide 19 by 4 and obtain remainder 3:

$$3 = 19 - (4 \cdot 4)$$

Finally

$$1 = 4 - 3$$

So 1 is the Greatest Common Divisor of 61 and 19. Substituting back into the above equations we obtain:

$$\begin{aligned} 1 &= 4 - 3 \\ &= 4 - (19 - 4 \cdot 4) = 5 \cdot 4 - 1 \cdot 19 \\ &= 5 \cdot (61 - 3 \cdot 19) - 1 \cdot 19 \\ &= -16 \cdot 19 + 5 \cdot 61 \\ &\equiv 45 \cdot 19 + 5 \cdot 61 \pmod{61} \\ &\equiv 45 \cdot 19 \pmod{61}. \end{aligned}$$

Thus, 19^{-1} modulo 61 is 45.

Now we state a beautiful theorem. If n is a positive integer, then a^n corresponds to $a \cdot a \cdot a \cdot \ldots \cdot a$ with n factors of a. We define $a^0 = 1$. So we also know how to take positive exponents.

Fermat's Little Theorem: If p is a prime integer, and p does not divide a, then $a^{p-1} \equiv 1 \pmod{p}$.

For example,

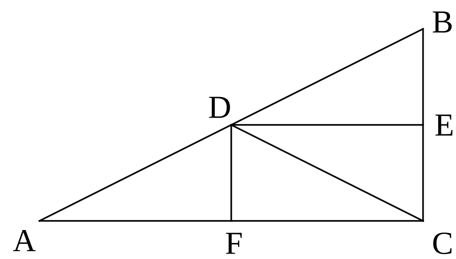
$$5^2 \equiv 1 \pmod{3} \qquad 2^6 \equiv 1 \pmod{7} \qquad 123456^{22} \equiv 1 \pmod{23} \qquad (-3)^{10} \equiv 1 \pmod{11}$$

This theorem is a key part of Luke Gustafson's proof above.

3/1/11. Triangle ABC has angle A measuring $30°$, angle B measuring $60°$, and angle C measuring $90°$. Show four different ways to divide triangle ABC into four triangles, each similar to triangle ABC, but with one quarter of the area. Prove that the angles and sizes of the smaller triangles are correct.

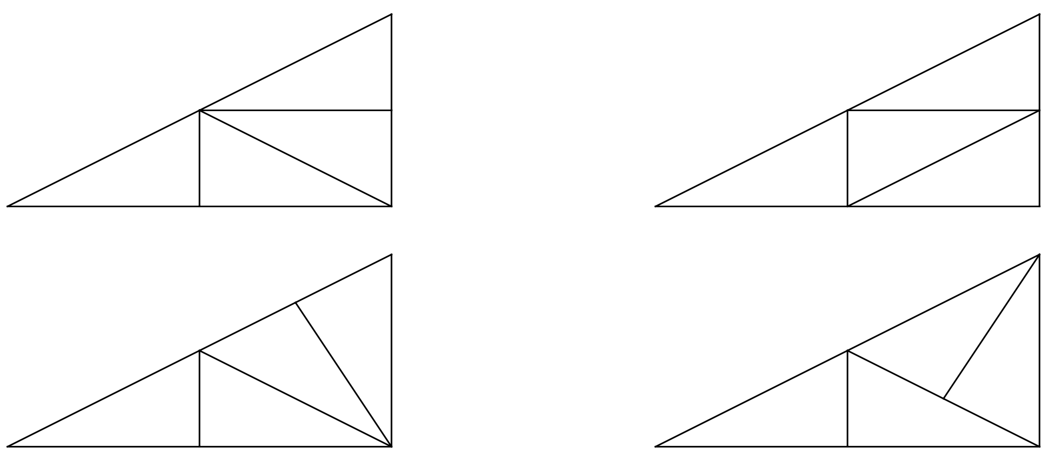
Solution 1 by Neeraj Kumar (12/NC): Let us start with triangle ABC having sides AB, BC, and AC of lengths 2, 1, and $\sqrt{3}$, respectively. The area of this triangle is $\frac{\sqrt{3}}{2}$. Therefore, each of the smaller triangles must have area $\frac{\sqrt{3}}{8}$ and sides of lengths 1, 1/2, and $\frac{\sqrt{3}}{2}$.

Here is the first way to divide ABC:
We choose E to be the midpoint of BC, and connect it to AB (parallel to AC). Since angle B is common to both triangles ABC and DBE, and E is also a right angle, the two triangles are similar. Also, since BE is 1/2, DE must equal $(\sqrt{3})/2$, and the area of DBE is indeed $\frac{\sqrt{3}}{8}$. If we now choose F to be the midpoint of AC, then again we create a similar situation with shared angle A of the two triangles ADF and ABC, and so ADF is also a valid smaller triangle. Now DC = DB = DA, by Side-Angle-Side congruence in the triangles BED and CED (BE = EC, angle BED = angle DEC, DE = DE), and in triangles ADF and CDF. Therefore, each of DEC and DFC is also a valid smaller triangle.



For the second way to divide the triangle, we can simply choose the other diagonal of rectangle DECF, i.e. FE rather than DC.

For the final two ways, we can see that triangle DBC is equilateral (because each side has length 1). Therefore, the altitude DE can be moved to any one of the three sides. Thus the four different ways are given here:



Editor's Comments: This problem, along with many earlier USAMTS problems, was proposed by Dr. Erin Schram of the National Security Agency. We are also thankful to him for his thoughtful evaluation of the problems considered for the program.

4/1/11. There are 8436 steel balls, each with radius 1 centimeter, stacked in a tetrahedral pile, with one ball on top, 3 balls in the second layer, 6 in the third layer, 10 in the fourth, and so on. Determine the height of the pile in centimeters.

Solution 1 by Shaili Jain (12/MI): The number of balls in each level of the pyramid are: *1, 3, 6,* *10, 15, 21, ..., n(n+1)/2.*

$$\sum_{i=1}^{n} \frac{i(i+1)}{2} = 8436$$

n = number of layers in the pyramid. We want to solve for n.

$$8436 = \sum_{i=1}^{n} \frac{i^2+i}{2} = \frac{1}{2}\left[\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i\right] = \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right]$$

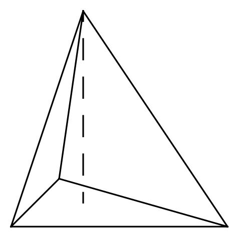
$$8436 = \frac{1}{6}n(n+1)(n+2) = \binom{n+2}{3}\text{, a binomial coefficient.}$$

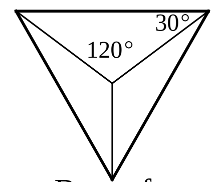
$$n = 36.$$

The sides of the tetrahedron are $35 \cdot 2 = 70$ cm each (they go through the centers of the outer balls).

$$\frac{\sin 120°}{70} = \frac{\sin 30°}{x} \text{ by Law of Sines}$$

$$x = \frac{70\sqrt{3}}{3}$$



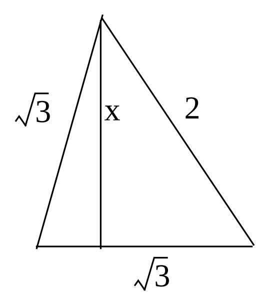


Base of tetrahedron

$$\text{Tetrahedron height} = \sqrt{70^2 - \left(\frac{70\sqrt{3}}{3}\right)^2} = \frac{70\sqrt{6}}{3}$$

The height of the pile of balls is $\frac{70\sqrt{6}}{3} + 2$ cm. because it is +1 cm. from the center of the top ball to the top, and +1 cm. from the center of a bottom ball to the bottom.

Solution 2 by Wendy Pang (11/CA): 8436 is the sum of 36 triangle numbers, meaning that there are 36 layers of balls. The centers of four adjacent balls form a tetrahedron (equilateral) with slant height of 2, which means that the height of the tetrahedron is $\frac{2\sqrt{6}}{3}$.



Height of pile = (layers - 1)(height of tetrahedron) + $2 \cdot$ radius.

$$= (36-1)\left(\frac{2\sqrt{6}}{3}\right) + 2(1)$$

$$= \frac{70\sqrt{6}}{3} + 2 \text{ cm.}$$

Editor's Comments: This problem was suggested 20 years ago by Professor Endre Hòdi, long-time leader of Hungary's successful teams to the IMO's.

5/1/11. In a convex pentagon ABCDE the sides have lengths 1, 2, 3, 4, and 5, though not necessarily in that order. Let F, G, H, and I be the midpoints of the sides AB, BC, CD, and DE, respectively. Let X be the midpoint of segment FH, and Y be the midpoint of segment GI. The length of segment XY is an integer. Find all possible values for the length of side AE.

Solution 1 by Melody Chan (12/NY): Use the notation $A = (4a_1, 4a_2)$ to mean "the coordinates of the point A are $(4a_1, 4a_2)$."

Let $A = (4a_1, 4a_2)$, $B = (4b_1, 4b_2)$, $C = (4c_1, 4c_2)$, $D = (4d_1, 4d_2)$, and $E = (4e_1, 4e_2)$.

Then
$$F = (2a_1+2b_1, 2a_2+2b_2)$$
$$G = (2b_1+2c_1, 2b_2+2c_2)$$
$$H = (2c_1+2d_1, 2c_2+2d_2)$$
$$I = (2d_1+2e_1, 2d_2+2e_2)$$

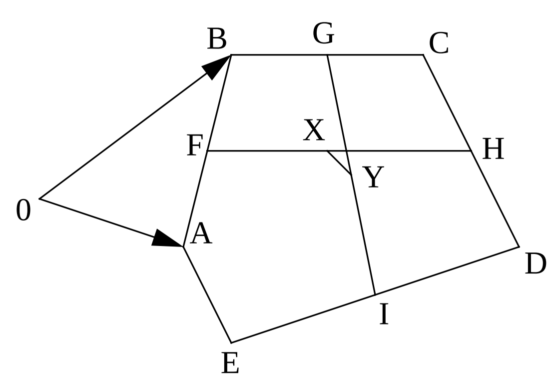
and
$$X = (a_1+b_1+c_1+d_1, a_2+b_2+c_2+d_2)$$
$$Y = (b_1+c_1+d_1+e_1, b_2+c_2+d_2+e_2)$$

Thus
$$XY = \sqrt{[(b_1+c_1+d_1+e_1)-(a_1+b_1+c_1+d_1)]^2+[(b_2+c_2+d_2+e_2)-(a_2+b_2+c_2+d_2)]^2}$$
$$= \sqrt{(e_1-a_1)^2+(e_2-a_2)^2}$$ which is given to be an integer.

Also
$$AE = \sqrt{(4e_1-4a_1)^2+(4e_2-4a_2)^2}$$
$$= 4\sqrt{(e_1-a_1)^2+(e_2-a_2)^2} = 4 \cdot XY$$

So AE is a multiple of 4. Thus $AE = 4$ is the only possible solution.

Solution 2 by Kartik Lamba (12/IL): The solution becomes simple when one uses vectors. Consider the pentagon shown at right with any of the given following dimensions:



$$F = \frac{A+B}{2} \qquad G = \frac{B+C}{2}$$

$$H = \frac{C+D}{2} \qquad I = \frac{D+E}{2}$$

$$X = \frac{\frac{A+B}{2}+\frac{C+D}{2}}{2} \qquad Y = \frac{\frac{B+C}{2}+\frac{D+E}{2}}{2}$$

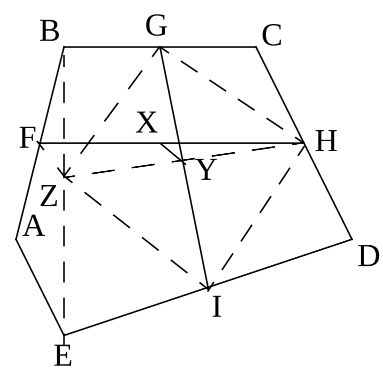
$$X - Y = \frac{A+B+C+D}{4} - \frac{B+C+D+E}{4} = \frac{A-E}{4}$$

$$4 \cdot \overline{XY} = \overline{AE}$$

In order for both lengths $\overline{XY}$ and $\overline{AE}$ to have integer values between 1 and 5, inclusive, $\overline{XY}$ must be 1 and $\overline{AE}$ must be 4. Thus

$$\overline{AE} = 4\,.$$

Solution 3 by Dmitry Portnyagin (12/NY): We draw line segment *BE*, and mark its midpoint *Z*. Now, *G* and *H* are midpoints of the sides of ΔBCD, *Z* and *I* are midpoints of the sides of ΔBED. Thus $\Delta GCH \sim \Delta BCD$, $\Delta ZEI \sim \Delta BED$, and in both cases the sides of the smaller triangle are half the size of the sides of the larger triangles. Thus, *ZI* = *GH* = *BD/2*, and $ZI \parallel GH \parallel BD$. Thus, *GHIZ* is a parallelogram. In a parallelogram the diagonals bisect each other. Thus *Y* is the midpoint of *HZ*.



F and *Z* are midpoints of *AB* and *BE*, respectively. Thus, once again, *FZ* = *AE/2*. *X* and *Y* are midpoints of *FH* and *HZ*, and once again *XY* = *FZ/2*. Thus, *XY* = *AE/4*. But *XY* has to be an integer, and out of the five possible lengths, only 4 is divisible by 4.

AE = 4

Editor's Comments: This clever problem was created for USAMTS by Professor Gregory Galperin, who is the author of a soon-to-be published compendium of problems for the famous Moscow Mathematical Olympiads.