U S A Mathematical Talent Search

PROBLEMS / SOLUTIONS / COMMENTS
Round 1 - Year 10 - Academic Year 1998-99

Gene A. Berg, Editor

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1/1/10. Several pairs of positive integers (m, n) satisfy the condition $19m + 90 + 8n = 1998$. Of these, $(100, 1)$ is the pair with the smallest value for n. Find the pair with the smallest value for m.

Solution 1 by Michael Castleman (12/MA): First we solve the given equation for n in terms of m, which gives $n = 238.5 - 2.375m$. Trying 1, 2, or 3 for m gives non-integer values for n, but trying 4 for m gives $n = 229$. Thus the answer is $(m, n) =$ **(4, 229)**.

Solution 2 by Amanda Felder (12/TX): First, subtract 90 from each side of the equation, so it reads $19m + 8n = 1908$. Since there is a remainder of 4 when 1908 is divided by 8, $19m$ must also have a remainder of 4 when divided by 8, if $8n$, and thus n, is to be an integer. Thus m must be divisible by four since $19m$ leaves a remainder of four when divided by 8. The smallest positive integer m divisible by 4 is 4. The pair with the smallest value for m is **(4, 229)**.

Solution 3 by Brian Cruz (11/TN): Rewrite the equation as $19m = 1908 - 8n$. Since n is a positive integer, this statement is equivalent to $19m \equiv 1908 \pmod 8$ given the restriction $0 < 19m < 1908$. It can be further reduced to $3m \equiv 4 \pmod 8$, or $m \equiv 4 \pmod 8$. The smallest value of m with this characteristic is, of course, 4. Therefore the pair of positive integers (m, n) satisfying $19m + 90 + 8n = 1998$ with the smallest value of m is **(4, 229)**.

Solution 4 by Blair Dowling (GA): Rewrite the equation as $19m + 8n = 1908$. First solve 19M + 8N = 1. Apply the extended Euclid's algorithm to inputs 19 and 8:

19 = 2x**8** + 3
8 = 2x3 + 2
3 = 1x2 + 1
2 = 2x1 + 0

Thus

19(3) + 8(-7) = 1

so M = 3 and N = -7.

"Magic Box" to generate a solution to 19M + 8N = 1.

		2	2	1	2
0	1	2	5	***7***	19
1	0	1	2	***3***	8

First solution to $19m + 18n = 1908$:

m = (M)(1908) = 3(1908) = 5724
n = (N)(1908) = (-7)(1908) = -13356.

We also have a solution to 19M' + 8N' = 0, namely M' = -8 and N' = 19.

So all solutions to $19m + 8n = 1908$ are given by $(m, n) = (5724, -13356) + a(-8, 19)$ for positive integer a. We want the solution with the smallest positive value for m.

Since $5724/8 = 715.5$, we set $a = 715$ to get

$(m, n) = (5724 - 8\text{x}715, -13356 + 19\text{x}715) = \mathbf{(4, 229)}$, our solution.

Solution 5 by Yen-Chieh Tseng (11/GA): This is a Diophantine equation of the type $Ax - By = D$, solvable using continued fractions. Converted into this form, the equation becomes $19m - 8(-n) = 1908$. Convert A/B into a continued fraction. The result is <2, 2, 1, 2> (see the editor's comment which follows for notation). Calculate the convergents, and then solve for the smallest possible value of m. The work is shown below:
First, the partial quotients, a_i, are given in the following table

i	-2	-1	0	1	2	3
a_i			2	2	1	2
p_i	0	1	2	5	7	19
q_i	1	0	1	2	3	8

Here, $p_i q_{i-1} - q_i p_{i-1} = (-1)^i$ for each integer i.
Set $M = Dq_2 + Bt = 1908 \cdot 3 + 8t$. With $0 < M$ we have $0 < 5724 + 8t$ and $-715.5 < t$. Similarly set $-N = Dp_2 + At$ so $N = -1908 \cdot 7 - 19t$. With $0 < N$ we have $0 < -13356 - 19t$ and $t < -702.9$.
Set $t = -715$, the smallest value possible, and obtain M=4 and N=229. So the integer pair (m, n) with the smallest value of m is **(4, 229)**. Checking these values in the original equation confirms the solution.

Editor's comments: Mr. Tseng's solution gives an opportunity to expand on two interesting mathematical topics, Diophantine equations and continued fractions.

The theory of Diophantine equations is concerned with integer solutions to polynomial equations. Consider, for example, the equation

$$x^2 - 2y^2 = \pm 1 .$$

Solutions (x, y) to this equation include (1, 1), (3, 2), and (7, 5).

Continued fractions yield solutions for certain types of Diophantine equations, including a type known as Pell's equations, which includes this equation. Let D be a positive real number greater than 1. Consider the following sequence of equations. In each equation a_i is a positive integer and $0 \le R_i < 1$. If $R_i = 0$ we stop; but if $R_i \neq 0$, then we can continue with the next equation. If

$0 < R_i < 1$, then we can write $R_i = \frac{1}{a_{i+1} + R_{i+1}}$, where a_{i+1} is a positive integer and $0 \le R_i < 1$.

$$\begin{aligned} D &= a_0 + R_0 \\ &= a_0 + \frac{1}{a_1 + R_1} \\ &= a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + R_2}} \\ &= a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}} = \langle a_0, a_1, a_2, a_3, \dots \rangle \end{aligned}$$

This is called the simple continued fraction expansion of D. Let

$$\frac{p_0}{q_0} = a_0 = \langle a_0 \rangle ,$$

$$\frac{p_1}{q_1} = a_0 + \frac{1}{a_1} = \langle a_0, a_1 \rangle .$$

In general let

$$\frac{p_i}{q_i} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\dots + \cfrac{1}{a_i}}} = \langle a_0, a_1, \dots, a_i \rangle$$

These rational numbers are called the convergents, and give increasingly accurate approximations to D.

The continued fraction expansion of $\sqrt{2}$ yields solutions to our equation, $x^2 - 2y^2 = \pm 1$. The continued fraction expansion of $\sqrt{2}$ is

$$\sqrt{2} = 1 + (\sqrt{2} - 1) = 1 + \frac{1}{2 + (\sqrt{2} - 1)} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}} = \langle 1, 2, 2, 2, \dots \rangle$$

A few of the convergents are given in the following table:

i	0	1	2	3	4	5	...
a_{i}	1	2	2	2	2	2	...
p_i	1	3	7	17	41	99	...
q_i	1	2	5	12	29	70	...
$\frac{p_i}{q_i}$	1	1.5	1.4	1.41666...	1.41379...	1.41428...	...

First, observe that the convegents are increasingly good estimates of $\sqrt{2}$, alternating between too large and too small. Second, observe that $p_i^2 - 2q_i^2 = \pm 1$ for each i. Thus there are an infinite number of solutions to $x^2 - 2y^2 = \pm 1$, and we are able to generate solutions very efficiently. Third, the similarities of the tables in solutions 4 and 5 reveal a close connection between the extended Euclidean algorithm and continued fractions.

2/1/10. Determine the smallest rational number $\frac{r}{s}$ such that $\frac{1}{k} + \frac{1}{m} + \frac{1}{n} \le \frac{r}{s}$ whenever k, m, and n are positive integers that satisfy the inequality $\frac{1}{k} + \frac{1}{m} + \frac{1}{n} < 1$.

Solution by Irena Foygel (10/IL): Without loss of generality, let $k \le m \le n$.

If $k = 2$: Then $m > 2$.

If $m = 3$: $\frac{1}{2} + \frac{1}{3} + \frac{1}{n} < 1$, $\frac{1}{n} < \frac{1}{6}$, $n > 6$. The minimum integer is $n = 7$.

$$\text{Maximum}\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{n}\right) = \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{7}\right) = \frac{41}{42}.$$

If $m = 4$: $\frac{1}{2} + \frac{1}{4} + \frac{1}{n} < 1$, $\frac{1}{n} < \frac{1}{4}$, $n > 4$. The minimum integer is $n = 5$.

$$\text{Maximum}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{n}\right) = \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{5}\right) = \frac{19}{20}.$$

If $m > 4$: $\frac{1}{2} + \frac{1}{m} + \frac{1}{n} < 1$, $\frac{1}{n} \le \frac{1}{m}$, $n > 4$. The minimum integer is $n = m$.

$$\text{Maximum}\left(\frac{1}{2} + \frac{1}{m} + \frac{1}{n}\right) = \left(\frac{1}{2} + \frac{1}{5} + \frac{1}{5}\right) = \frac{9}{10}.$$

If $k = 3$: Then $m > 2$.

If $m = 3$: $\frac{1}{3} + \frac{1}{3} + \frac{1}{n} < 1$, $\frac{1}{n} < \frac{1}{3}$, $n > 3$. The minimum integer is $n = 4$.

$$\text{Maximum}\left(\frac{1}{3} + \frac{1}{3} + \frac{1}{n}\right) = \left(\frac{1}{3} + \frac{1}{3} + \frac{1}{4}\right) = \frac{11}{12}.$$

If $m > 3$: $\frac{1}{3} + \frac{1}{m} + \frac{1}{n} < 1$, $\frac{1}{n} \le \frac{1}{m}$, $n \ge m$. The minimum integer is $n = m$.

$$\text{Maximum}\left(\frac{1}{3} + \frac{1}{m} + \frac{1}{n}\right) = \left(\frac{1}{3} + \frac{1}{m} + \frac{1}{m}\right) = \frac{5}{6}.$$

If $k > 3$: $\frac{1}{k}+\frac{1}{m}+\frac{1}{n}<1$, $m \geq k$, $n \geq k$. For minimum m, $m = k$. For minimum n, $n = k$.

$$\text{Maximum}\left(\frac{1}{k}+\frac{1}{m}+\frac{1}{n}\right) = \left(\frac{1}{k}+\frac{1}{k}+\frac{1}{k}\right) = \frac{3}{4} \quad .$$

In summary, for $\frac{1}{k}+\frac{1}{m}+\frac{1}{n}<1$, k, m, and n positive integers, the maximum$\left(\frac{1}{k}+\frac{1}{m}+\frac{1}{n}\right) = \frac{41}{42}$.

So $\frac{r}{s} = \frac{41}{42}$.

3/1/10. It is possible to arrange eight of the nine numbers
2, 3, 4, 7, 10, 11, 12, 13, 15
in the vacant squares of the 3 by 4 array shown on the right so that the arithmetic average of the numbers in each row and in each column is the same integer. Exhibit such an arrangement, and specify which one of the nine numbers must be left out when completing the array.

1			
	9		5
		14	

Solution by Raymond Chen (10/CA): We are given 13 integers (1 through 15 except 6 and 8), of which we must use twelve to fill the array. Since the average of the numbers in each row and column is an integer [given], and there are 3 rows and 4 columns, the sum of the 12 numbers must be divisible by 3 and 4; therefore the sum must be divisible by 12. Since the sum of the thirteen numbers given is $106 \equiv 10 \pmod{12}$, we must **leave out "10"** from our 12 numbers. The sum of the remaining numbers is 96, so the sum for each row is 32 = 96/3 and the sum for each column is 24 = 96/4. Thus,

1	A	B	C
D	9	E	5
F	G	14	H

(a) D + E = 18, so D and E must be 3 and 15 or 7 and 11 in some order.
(b) D + F = 23, so D and F must be 11 and 12, in some order.

From (a) and (b), D = 11, so F = 12 and E = 7, from which B = 3.

Now,
(c) G + H = 6, so G and H must be 2 and 4 in some order.
(d) A + G = 15, so A and G must be 2 and 13 in some order.

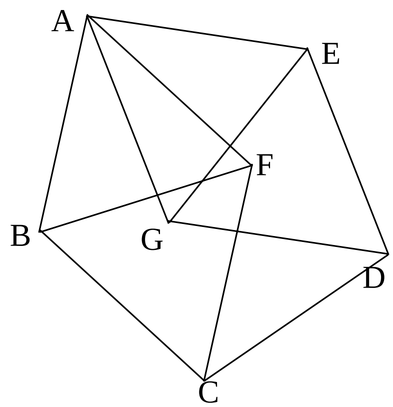
From (c) and (d), G = 2, so H = 4 and A = 13, from which C = 15.

1	13	3	15
11	9	7	5
12	2	14	4

Therefore, the desired array is shown on the right, with "10" left out of the given nine numbers.

4/1/10. Show that it is possible to arrange seven distinct points in the plane so that among any three of these seven points, two of the points are a unit distance apart. (Your solution should include a carefully prepared sketch of the seven points, along with all segments that are of unit length.)

Solution by Michael Seeman (11/CA): See the diagram at the right. All marked segments have a length of one unit. Note that it is not a regular pentagon since not all angles are congruent.



Requirement #1: Given any three points chosen from the figure, two are a unit apart.

(a) A pentagon with unit sides meets requirement #1, because it is impossible to divide 5 into 3 groups of 2 (or more). (Note the angles are not congruent.)

(b) Placing point F one unit away from points A, B, and C, still meets requirement #1, because the three points including F and either A, B, or C, have a distance one unit between F and either A, B, or C. The only three points remaining are F, E, and D, in which E and D are one unit apart.

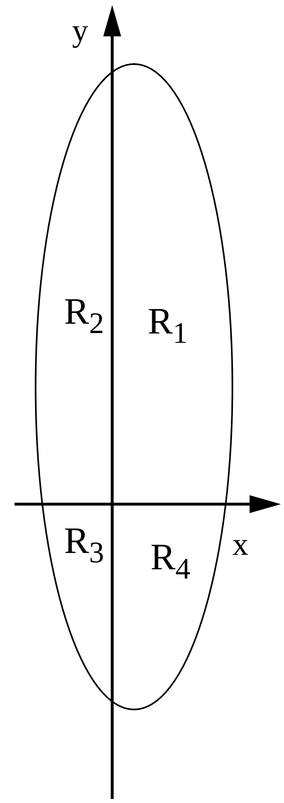
(c) Use similar reasoning for the placement of point G. The three remaining possibilities, with a unit length segment, are:

Three Points	Unit Length Segment
G, B C	BC
G, B, F	BF
G, C, F	CF.

(d) All possible sets of three points have two that are a unit apart, so this figure of seven points meets requirement #1.

5/1/10. The figure on the right shows the ellipse

$$\frac{(x-19)^2}{19} + \frac{(y-98)^2}{98} = 1998.$$



Let R_1, R_2, R_3, and R_4 denote those areas within the ellipse that are in the first, second, third, and fourth quadrants, respectively. Determine the value of $R_1 - R_2 + R_3 - R_4$.

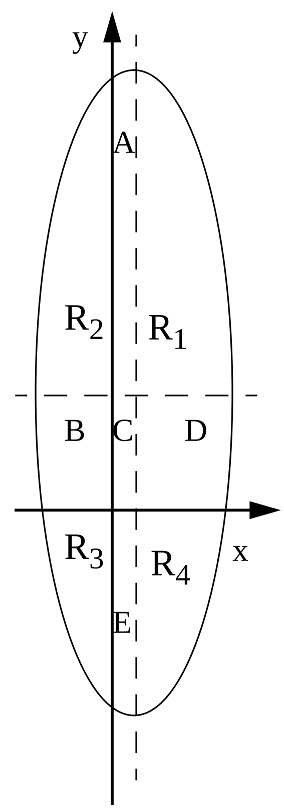
Solution by Alex Wissner-Gross (12/NY): The ellipse may be expressed in the form

$$\frac{(x-19)^2}{19 \bullet 1998} + \frac{(y-98)^2}{98 \bullet 1998} = 1 \quad .$$

In this form it is clear the ellipse is centered at (19, 98) and has lines of both horizontal and vertical symmetry that run through its center (shown with dashed lines on the following diagram).

If the regions of the ellipse between these lines of symmetry and the x and y axes are labeled A, B, C, D, and E as shown in the diagram, and if the total area of the ellipse is denoted by S, then R_1, R_2, R_3, and R_4 may be expressed

$$R_1 = S/4 + A + C + D$$
$$R_2 = S/4 - A + B$$
$$R_3 = S/4 - B - C - E$$
$$R_4 = S/4 - D + E.$$



$R_1 - R_2 + R_3 - R_4$ may thus be expressed:

$$(S/4 + A + C + D) - (S/4 - A + B) + (S/4 - B - C - E) - (S/4 - D + E)$$

$$= 2A - 2B + 2D - 2E.$$

Because of vertical and horizontal symmetry,

$$A = C + E$$

and

$$D = B + C.$$

Therefore, $2A - 2B + 2D - 2E =$

$$=2(C + E) - 2B + 2(B + C) - 2E$$
$$=4C.$$

Since C is bounded by the axes ($x = 0$ and $y = 0$) and the lines $x = 19$ and $y = 98$, C has an area of $19 \bullet 98 = 1862$.

Therefore the area of $R_1 - R_2 + R_3 - R_4$ is **4C = 7448 units^2.**

Editor's Comment: During the summer of 1998, Mr. Wissner-Gross represented the United States at the International Olympiad in Informatics (IOI '98), after having been ranked number 2 in America at the USA Computer Olympics training camp. I have said before that many USAMTS alumni go on to represent the United States in the various science and mathematics Olympiads. Although Alex is not yet an alumnus, he has proven me correct again.