USA Mathematical Talent Search

Round 2 Problems

Year 22 — Academic Year 2010–2011

www.usamts.org

Important information:

1. **You must show your work and prove your answers on all problems.** If you just send a numerical answer for a problem with no proof, you will get no more than 1 point.

2. Put your name and USAMTS ID# on **every page you submit**.

3. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page, and you should only submit work on one side of each piece of paper.

4. Submit your solutions by January 24, 2011, via one (and only one!) of the methods below:

   (a) Web: Log on to `www.usamts.org` to upload a PDF file containing your solutions. (No other file type will be accepted.)
   Deadline: 3 PM Eastern / Noon Pacific on January 24

   (b) Mail: USAMTS, P.O. Box 2090, Alpine, CA 91903–2090.
   (Solutions must be postmarked on or before January 24.)

5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.

6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging onto `www.usamts.org` and visiting the "My USAMTS" pages.

7. Round 2 results will be posted at `www.usamts.org` when available. To see your results, log on to the USAMTS website, then go to "My USAMTS". You will also receive an email when your scores and comments are available (provided that you did item #6 above).

These are only part of the complete rules.
Please read the entire rules on `www.usamts.org`.

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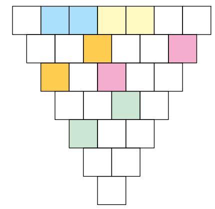
Each problem is worth 5 points.

1/2/22. Show that there is a unique way to place positive integers in the grid to the right following these three rules:



1. Each entry in the top row is one digit.
2. Each entry in any row below the top row is the sum of the two entries immediately above it.
3. Each pair of same-color squares contain the same integer. These five distinct integers are used exactly twice and no other integer is used more than once.

(You can find a larger labeled version of this diagram on the back page.)

2/2/22. A sequence is called **tworrific** if its first term is 1 and the sum of every pair of consecutive terms is a positive power of 2. One example of a tworrific sequence is 1, 7, -5, 7, 57.

(a) Find the shortest possible length of a tworrific sequence that contains the term 2011.

(b) Find the number of tworrific sequences that contain the term 2011 and have this shortest possible length.

3/2/22. Richard, six of his friends, and a Gortha beast are standing at different vertices of a cube-shaped planet. Richard has a potato and is a neighbor to the Gortha. On each turn, whoever has the potato throws it at random to one of his three neighbors. If the Gortha gets the potato he eats it. What is the probability that Richard is the one who feeds the Gortha?

4/2/22. Let A, B, C, and D be points in the plane such that $AD \parallel BC$. Let I be the incenter of $\triangle ABC$ and assume that I is also the orthocenter of $\triangle DBC$. Show that $AB + AC = 2BC$.

5/2/22. Zara and Ada are playing a game. Ada begins by picking an integer from 1 to 2011 (inclusive). On each turn Zara tries to guess Ada's number. Ada then tells Zara whether her guess is too high, too low, or correct. If Zara's guess is not correct, Ada adds or subtracts 1 from her number (always constructing a new number from 1 to 2011). Assuming Zara plays optimally, what is the minimum number of turns she needs to guarantee that she will guess Ada's number?

6/2/22. The roving rational robot rolls along the rational number line. On each turn, if the robot is at $\frac{p}{q}$, he selects a positive integer n and rolls to $\frac{p+nq}{q+np}$. The robot begins at the rational number 2011. Can the roving rational robot ever reach the rational number 2?

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