USA Mathematical Talent Search

Round 1 Problems

Year 22 — Academic Year 2010–2011

www.usamts.org

Important information:

1. **You must show your work and prove your answers on all problems.** If you just send a numerical answer for a problem with no proof, you will get no more than 1 point.

2. Put your name and USAMTS ID# on **every page you submit**.

3. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page, and you should only submit work on one side of each piece of paper.

4. Submit your solutions by November 22, 2010, via one (and only one!) of the methods below:

   (a) Web: Log on to `www.usamts.org` to upload a PDF file containing your solutions. (No other file type will be accepted.)
   Deadline: 3 PM Eastern / Noon Pacific on November 22

   (b) Mail: USAMTS, P.O. Box 2090, Alpine, CA 91903–2090.
   (Solutions must be postmarked on or before November 22.)

5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.

6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging onto `www.usamts.org` and visiting the "My USAMTS" pages.

7. Round 1 results will be posted at `www.usamts.org` when available. To see your results, log on to the USAMTS website, then go to "My USAMTS". You will also receive an email when your scores and comments are available (provided that you did item #6 above).

These are only part of the complete rules.
Please read the entire rules on `www.usamts.org`.

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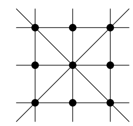
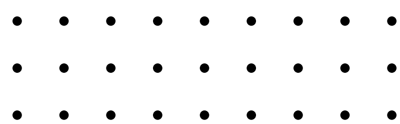
Each problem is worth 5 points.

1/1/22.

Given a set S of points in the plane, a line is called *happy* if it contains at least 3 points in S. For example, if S is the 3×3 grid of points shown at right, then there are 8 happy lines as shown.



(a) If S is the 3×9 grid shown below, how many happy lines are there?



(b) Find, with proof, a set S (in the plane) with 27 points that has exactly 49 happy lines.

2/1/22. You're at vertex A of triangle ABC, where $\angle B = \angle C = 65^\circ$. The sides of the triangle are perfectly reflective; if you shoot a laser from A to the midpoint of $\overline{BC}$, it will reflect once and return to A. Suppose you fire at a point on $\overline{BC}$ other than its midpoint, and the beam still returns to A after reflecting some number of times. What is the smallest number of reflections the beam can make before returning to A? What is the smallest angle between $\overline{AB}$ and the initial beam that produces this number of reflections?

3/1/22. Find $c > 0$ such that if r, s, and t are the roots of the cubic

$$f(x) = x^3 - 4x^2 + 6x + c,$$

then

$$1 = \frac{1}{r^2 + s^2} + \frac{1}{s^2 + t^2} + \frac{1}{t^2 + r^2}.$$

4/1/22. Sasha has a compass with fixed radius s and Rebecca has a compass with fixed radius r. Sasha draws a circle (with his compass) and Rebecca then draws a circle (with her compass) that intersects Sasha's circle twice. We call these intersection points C and D.

Charlie draws a common tangent to both circles, meeting Sasha's circle at point A and Rebecca's circle at point B, and then draws the circle passing through A, B, and C. Prove that the radius of Charlie's circle does not depend on where Sasha and Rebecca choose to draw their circles, or which of the two common tangents Charlie draws.

5/1/22. A convex polygon P is called *peculiar* if: (a) for some $n \geq 3$, the vertices of P are a subset of the vertices of a regular n-gon with sides of length 1; (b) the center O of the n-gon lies outside of P; and (c) for every integer k with $0 < k \leq \frac{n}{2}$, the quantity $\frac{2k\pi}{n}$ is the measure of exactly one $\angle AOB$, where A and B are vertices of P. Find the number of non-congruent peculiar polygons.

6/1/22. There are 50 people (numbered 1 to 50) and 50 identically wrapped presents around a table at a party. Each present contains an integer dollar amount from \$1 to \$50, and no two presents contain the same amount. Each person is randomly given one of the presents. Beginning with player #1, each player in turn does **one** of the following:

1. Opens his present and shows everyone the contents; **or**

2. If another player at the table has an open present, the player whose turn it is may swap presents with that player, and leave the table with the open present. The other player then immediately opens his new present and shows everyone the contents.

For example, the game could begin as follows:

- Player #1 opens his present. (The game must *always* begin this way, as there are no open presents with which to swap.)
- Player #2 decides to swap her present with Player #1. Player #2 takes the money from her newly acquired present and leaves the table. Player #1 opens his new present (which used to belong to Player #2).
- Player #3 opens her present. (Now Players #1 and #3 have open presents, and Player #2 is still away from the table.)
- Player #4 decides to swap his present with Player #1. Player #4 takes the money from his newly acquired present and leaves the table. Player #1 opens his new present (which used to belong to Player #4).

The game ends after all the presents are opened, and all players keep the money in their currently held presents.

Suppose each player follows a strategy that maximizes the expected value that the player keeps at the end of the game.

(a) Find, with proof, the strategy each player follows. That is, describe when each player will choose to swap presents with someone, or keep her original present.

(b) What is the expected number of swaps?