USA Mathematical Talent Search

Round 2 Problems

Year 21 — Academic Year 2009–2010

www.usamts.org

Important information:

1. **You must show your work and prove your answers on all problems.** If you just send a numerical answer for a problem with no proof, you will get no more than 1 point.

2. Put your name and USAMTS ID# on **every page you submit**.

3. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page, and you should only submit work on one side of each piece of paper.

4. Submit your solutions by November 23, 2009, via one (and only one!) of the methods below:

   (a) Web: Log on to www.usamts.org to upload a PDF file containing your solutions. (No other file type will be accepted.)
   Deadline: 3 PM Eastern / Noon Pacific on November 23

   (b) Mail: USAMTS, P.O. Box 2090, Alpine, CA 91903–2090.
   (Solutions must be postmarked on or before November 23.)

5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.

6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging onto www.usamts.org and visiting the "My USAMTS" pages. (If you are registered for the USAMTS and haven't received any email from us about the USAMTS, then your email address is probably wrong in your Profile.)

7. Round 2 results will be posted at www.usamts.org when available. To see your results, log on to the USAMTS website, then go to "My USAMTS". You will also receive an email when your scores and comments are available (provided that you did item #6 above).

These are only part of the complete rules.
Please read the entire rules on www.usamts.org.

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Each problem is worth 5 points.

1/2/21. Jeremy has a magic scale, each side of which holds a positive integer. He plays the following game: each turn, he chooses a positive integer n. He then adds n to the number on the left side of the scale, and multiplies by n the number on the right side of the scale. (For example, if the turn starts with 4 on the left and 6 on the right, and Jeremy chooses $n = 3$, then the turn ends with 7 on the left and 18 on the right.) Jeremy wins if he can make both sides of the scale equal.

(a) Show that if the game starts with the left scale holding 17 and the right scale holding 5, then Jeremy can win the game in 4 or fewer turns.

(b) Prove that if the game starts with the right scale holding b, where $b \geq 2$, then Jeremy can win the game in $b - 1$ or fewer turns.

2/2/21. Alice has three daughters, each of whom has two daughters; each of Alice's six granddaughters has one daughter. How many sets of women from the family of 16 can be chosen such that no woman and her daughter are both in the set? (Include the empty set as a possible set.)

3/2/21. Prove that if a and b are positive integers such that $a^2 + b^2$ is a multiple of 7^{2009}, then ab is a multiple of 7^{2010}.

4/2/21. The Rational Unit Jumping Frog starts at $(0, 0)$ on the Cartesian plane, and each minute jumps a distance of exactly 1 unit to a point with rational coordinates.

(a) Show that it is possible for the frog to reach the point $\left(\frac{1}{5}, \frac{1}{17}\right)$ in a finite amount of time.

(b) Show that the frog can never reach the point $\left(0, \frac{1}{4}\right)$.

5/2/21. Let ABC be a triangle with $AB = 3$, $AC = 4$, and $BC = 5$, let P be a point on $\overline{BC}$, and let Q be the point (other than A) where the line through A and P intersects the circumcircle of ABC. Prove that

$$PQ < \frac{25}{4\sqrt{6}}.$$

Round 2 Solutions must be submitted by **November 23, 2009**.

Please visit **http://www.usamts.org** for details about solution submission.