USA Mathematical Talent Search

Round 2 Problems

Year 20 — Academic Year 2008–2009

`www.usamts.org`

Important information:

1. **You must show your work and prove your answers on all problems.** If you just send a numerical answer for a problem with no proof, you will get no more than 1 point.
2. Put your name and USAMTS ID# on **every page you submit**.
3. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page, and you should only submit work on one side of each piece of paper.
4. Submit your solutions by November 24, 2008, via one (and only one!) of the methods below:
   (a) Web: Log on to `www.usamts.org` to upload a PDF file containing your solutions. (No other file type will be accepted.)
   (b) Mail: USAMTS, P.O. Box 2090, Alpine, CA 91903–2090. (Solutions must be postmarked on or before the deadline.)
5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.
6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging onto `www.usamts.org` and visiting the "My USAMTS" pages. (If you are registered for the USAMTS and haven't received any email from us about the USAMTS, then your email address is probably wrong in your Profile.)
7. Round 2 results will be posted at `www.usamts.org` when available. To see your results, log on to the USAMTS website, then go to "My USAMTS". You will also receive an email when your scores and comments are available (provided that you did item #6 above).

These are only part of the complete rules.
Please read the entire rules on `www.usamts.org`.

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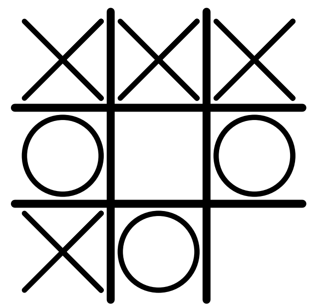
Each problem is worth 5 points.



1/2/20. Sarah and Joe play a standard 3-by-3 game of tic-tac-toe. Sarah goes first and plays X, and Joe goes second and plays O. They alternate turns placing their letter in an empty space, and the first to get 3 of their letters in a straight line (across, down, or diagonal) wins. How many possible final positions are there, given that Sarah wins on her 4th move? (Don't assume that the players play with any sort of strategy; one example of a possible final position is shown at right.)

2/2/20. Let C_1, C_2, C_3, and C_4 be four circles, with radii 1, 1, 3, and 3, respectively, such that circles C_1 and C_2, C_2 and C_3, C_3 and C_4, and C_4 and C_1 are externally tangent. A fifth circle C is smaller than the four other circles and is externally tangent to each of them. Find the radius of C.

3/2/20. Find, with proof, all polynomials $p(x)$ with the following property:

There exists a sequence $a_0, a_1, a_2, \ldots$ of positive integers such that $p(a_0) = 1$ and for all positive integers n:

(a) $p(a_n)$ is a positive integer, and

(b) $\displaystyle\sum_{j=0}^{n-1} \frac{1}{a_j} + \frac{1}{p(a_n)} = 1.$

4/2/20. Find, with proof, the largest positive integer k with the following property:

There exists a positive number N such that N is divisible by all but three of the integers $1, 2, 3, \ldots, k$, and furthermore those three integers (that don't divide N) are consecutive.

5/2/20. The set S consists of 2008 points evenly spaced on a circle of radius 1 (so that S forms the vertices of a regular 2008-sided polygon). 3 distinct points X, Y, Z in S are chosen at random. The expected value of the area of $\triangle XYZ$ can be written in the form $r \cot\left(\frac{\pi}{2008}\right)$, where r is a rational number. Find r.

Round 2 Solutions must be submitted by **November 24, 2008**.
Please visit **http://www.usamts.org** for details about solution submission.