# USA Mathematical Talent Search
Round 4 Problems
Year 19 — Academic Year 2007–2008
www.usamts.org

**Please follow the rules below:**

1. You must show your work and prove your answers on all problems. If you just send a numerical answer for a problem with no proof, you will get no more than 1 point.
2. If you have not already sent an Entry Form, download an Entry Form from the Forms page at

   `http://www.usamts.org/MyUSAMTS/U_MyForms.php`

   and submit the completed form with your solutions.
3. If you have already sent in an Entry Form and a Permission Form, you do not need to resend them.
4. Put your name and USAMTS ID# on **every page you submit**.
5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.
6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging into www.usamts.org, then clicking on My USAMTS on the sidebar, then clicking Profile. If you are registered for the USAMTS and haven't received any email from us about the USAMTS, your email address is probably wrong in your Profile.
7. Do not fax solutions written in pencil.
8. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page, and you should only submit work on one side of each piece of paper.
9. Results will be posted at www.usamts.org by mid-April. To see your results, log in to the USAMTS page, then go to My USAMTS. Check that your email address in your USAMTS Profile is correct; you will receive an email when the scores are available.
10. Submit your solutions by March 11, 2008 (postmark deadline), via one (and only one!) of the methods below.
    (a) Email: solutions@usamts.org. Please see usamts.org for a list of acceptable file types. Do not send Microsoft Word files.
    (b) Fax: (619) 445-2379 (You must include a cover sheet indicating the number of pages you are faxing, your name, and your USAMTS ID#.)
    (c) Snail mail: USAMTS, P.O. Box 2090, Alpine, CA 91903–2090.
11. Re–read Items 1–10.

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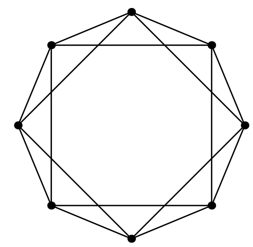
**1/4/19.** In the diagram at right, each vertex is labeled with a different positive factor of 2008, such that if two vertices are connected by an edge, then the label of one vertex divides the label of the other vertex. In how many different ways can the vertices be labeled? Two labelings are considered the same if one labeling can be obtained by rotating and/or reflecting the other labeling.



**2/4/19.** Determine, with proof, the greatest integer $n$ such that

$$\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n}{11} \right\rfloor + \left\lfloor \frac{n}{13} \right\rfloor < n,$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to $x$.

**3/4/19.** Let $0 < \mu < 1$. Define a sequence $\{a_n\}$ of real numbers by $a_1 = 1$ and for all integers $k \geq 1$,

$$\begin{aligned} a_{2k} &= \mu a_k, \\ a_{2k+1} &= (1-\mu) a_k. \end{aligned}$$

Find the value of the sum $\displaystyle\sum_{k=1}^{\infty} a_{2k} a_{2k+1}$ in terms of $\mu$.

**4/4/19.** Suppose that $w, x, y, z$ are positive real numbers such that $w + x < y + z$. Prove that it is impossible to simultaneously satisfy both

$$(w+x)yz < wx(y+z) \qquad \text{and} \qquad (w+x)(y+z) < wx + yz.$$

(Continued on next page.)

**5/4/19.** Let $P_1P_2P_3\cdots P_{13}$ be a regular 13-gon. For $1 \le i \le 6$, let $d_i = P_1P_{i+1}$. The 13 diagonals of length $d_6$ enclose a smaller regular 13-gon, whose side length we denote by $s$. Express $s$ in the form

$$s = c_1d_1 + c_2d_2 + c_3d_3 + c_4d_4 + c_5d_5 + c_6d_6,$$

where $c_1$, $c_2$, $c_3$, $c_4$, $c_5$, and $c_6$ are integers.