USA Mathematical Talent Search
Round 4 Problems
Year 18 — Academic Year 2006–2007
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Please follow the rules below to ensure that your paper is graded properly.

1. You must show your work and prove your answers on all problems. If you just send a numerical answer for a problem with no proof, you will get no more than 1 point.
2. If you have not already sent an Entry Form, download an Entry Form from the Forms page at

   `http://www.usamts.org/MyUSAMTS/U_MyForms.php`

   and submit the completed form with your solutions.
3. If you have already sent in an Entry Form and a Permission Form, you do not need to resend them.
4. Put your name and USAMTS ID# on **every page you submit**.
5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.
6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging into the site, then clicking on My USAMTS on the sidebar, then click Profile. If you are registered for the USAMTS and haven't received any email from us about the USAMTS, your email address is probably wrong in your Profile.
7. Do not fax solutions written in pencil.
8. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page.
9. By the end of October, Round 1 results will be posted at www.usamts.org. To see your results, log in to the USAMTS page, then go to My USAMTS. Check that your email address in your USAMTS Profile is correct; you will receive an email when the scores are available.
10. Submit your solutions by March 12, 2007 (postmark deadline), via one (and only one!) of the methods below.
    (a) Email: solutions@usamts.org. Please see usamts.org for a list of acceptable file types. Do not send .doc Microsoft Word files.
    (b) Fax: (619) 445-2379 (Please include a cover sheet indicating the number of pages you are faxing, your name, and your User ID.)
    (c) Snail mail: USAMTS, P.O. Box 2090, Alpine, CA 91903–2090.
11. Re–read Items 1–10.

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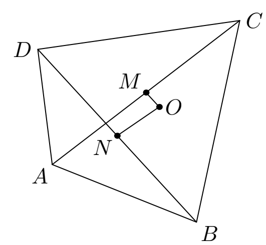
1/4/18. Let $S(n) = \sum_{i=1}^{n} (-1)^{i+1} i$. For example, $S(4) = 1 - 2 + 3 - 4 = -2$.

(a) Find, with proof, all positive integers a, b such that $S(a) + S(b) + S(a + b) = 2007$.
(b) Find, with proof, all positive integers c, d such that $S(c) + S(d) + S(c + d) = 2008$.

2/4/18. For how many integers n between 1 and 10^{2007}, inclusive, are the last 2007 digits of n and n^3 the same? (If n or n^3 has fewer than 2007 digits, treat it as if it had zeros on the left to compare the last 2007 digits.)

3/4/18. Let $ABCD$ be a convex quadrilateral. Let M be the midpoint of diagonal $\overline{AC}$ and N be the midpoint of diagonal $\overline{BD}$. Let O be the intersection of the line through N parallel to $\overline{AC}$ and the line through M parallel to $\overline{BD}$. Prove that the line segments joining O to the midpoints of each side of $ABCD$ divide $ABCD$ into four pieces of equal area.



4/4/18. We are given a $2 \times n$ array of nodes, where n is a positive integer. A *valid connection* of the array is the addition of 1-unit-long horizontal and vertical edges between nodes, such that each node is connected to every other node via the edges, and there are no loops of any size. We give some examples for $n = 3$:

Valid | Valid | Invalid: loop | Invalid: not connected

Let T_n denote the number of valid connections of the $2 \times n$ array. Find T_{10}.

5/4/18. A sequence of positive integers $(x_1, x_2, \ldots, x_{2007})$ satisfies the following two conditions:

(1) $x_n \neq x_{n+1}$ for $1 \leq n \leq 2006$, and

(2) $A_n = \dfrac{x_1 + x_2 + \cdots + x_n}{n}$ is an integer for $1 \leq n \leq 2007$.

Find the minimum possible value of A_{2007}.

Round 4 Solutions must be submitted by **March 12, 2007**.
Please visit **http://www.usamts.org** for details about solution submission.