USA Mathematical Talent Search

Round 3 Problems

Year 18 — Academic Year 2006–2007

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Please follow the rules below to ensure that your paper is graded properly.

1. You must show your work and prove your answers on all problems. If you just send a numerical answer for a problem with no proof, you will get no more than 1 point.

2. If you have not already sent an Entry Form, download an Entry Form from the Forms page at

   `http://www.usamts.org/MyUSAMTS/U_MyForms.php`

   and submit the completed form with your solutions.

3. If you have already sent in an Entry Form and a Permission Form, you do not need to resend them.

4. Put your name and USAMTS ID# on **every page you submit**.

5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.

6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging into the site, then clicking on My USAMTS on the sidebar, then click Profile. If you are registered for the USAMTS and haven't received any email from us about the USAMTS, your email address is probably wrong in your Profile.

7. Do not fax solutions written in pencil.

8. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page.

9. By the end of December, Round 2 results will be posted at www.usamts.org. To see your results, log in to the USAMTS page, then go to My USAMTS. Check that your email address in your USAMTS Profile is correct; you will receive an email when the scores are available.

10. Submit your solutions by January 8, 2007 (postmark deadline), via one (and only one!) of the methods below.

    (a) Email: solutions@usamts.org. Please see usamts.org for a list of acceptable file types. Do not send .doc Microsoft Word files.

    (b) Fax: (619) 445-2379 (Please include a cover sheet indicating the number of pages you are faxing, your name, and your User ID.)

    (c) Snail mail: USAMTS, P.O. Box 2090, Alpine, CA 91903–2090.

11. Re–read Items 1–10.

1/3/18. In how many distinguishable ways can the edges of a cube be colored such that each edge is yellow, red, or blue, and such that no two edges of the same color share a vertex? (Two cubes are indistinguishable if they can be rotated into positions such that the two cubes are colored exactly the same.)

2/3/18. Find, with proof, all real numbers x between 0 and 2π such that

$$\tan 7x - \sin 6x = \cos 4x - \cot 7x.$$

3/3/18. Three circles with radius 2 are drawn in a plane such that each circle is tangent to the other two. Let the centers of the circles be points A, B, and C. Point X is on the circle with center C such that $AX + XB = AC + CB$. Find the area of $\triangle AXB$.

4/3/18. Alice plays in a tournament in which every player plays a game against every other player exactly once. In each game, either one player wins and earns 2 points while the other gets 0 points, or the two players tie and both players earn 1 point. After the tournament, Alice tells Bob how many points she earned. Bob was not in the tournament, and does not know what happened in any individual game of the tournament.

(a) Suppose there are 12 players in the tournament, including Alice. What is the smallest number of points Alice could have earned such that Bob can deduce that Alice scored more points than at least 8 other players?

(b) Suppose there are n players in the tournament, including Alice, and that Alice scored m points. Find, in terms of n and k, the smallest value of m such that Bob can deduce that Alice scored more points than at least k other players.

5/3/18. Let $f(x)$ be a strictly increasing function defined for all $x > 0$ such that $f(x) > -\frac{1}{x}$ and $f(x)f(f(x) + \frac{1}{x}) = 1$ for all $x > 0$. Find $f(1)$.