USA Mathematical Talent Search

Round 2 Problems

Year 18 — Academic Year 2006–2007

www.usamts.org

Please follow the rules below to ensure that your paper is graded properly.

1. You must show your work and prove your answers on all problems. If you just send a numerical answer for a problem with no proof, you will get no more than 1 point.
2. If you have not already sent an Entry Form, download an Entry Form from the Forms page at

   `http://www.usamts.org/MyUSAMTS/U_MyForms.php`

   and submit the completed form with your solutions.
3. If you have already sent in an Entry Form and a Permission Form, you do not need to resend them.
4. Put your name and USAMTS ID# on **every page you submit**.
5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.
6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging into the site, then clicking on My USAMTS on the sidebar, then click Profile. If you are registered for the USAMTS and haven't received any email from us about the USAMTS, your email address is probably wrong in your Profile.
7. Do not fax solutions written in pencil.
8. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page.
9. By the end of October, Round 1 results will be posted at www.usamts.org. To see your results, log in to the USAMTS page, then go to My USAMTS. Check that your email address in your USAMTS Profile is correct; you will receive an email when the scores are available.
10. Submit your solutions by November 20, 2006 (postmark deadline), via one (and only one!) of the methods below.
    (a) Email: solutions@usamts.org. Please see usamts.org for a list of acceptable file types. Do not send .doc Microsoft Word files.
    (b) Fax: (619) 445-2379 (Please include a cover sheet indicating the number of pages you are faxing, your name, and your User ID.)
    (c) Snail mail: USAMTS, P.O. Box 2090, Alpine, CA 91903–2090.
11. Re–read Items 1–10.

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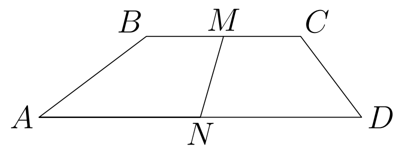
1/2/18. Find all positive integers n such that the sum of the squares of the digits of n is 2006 less than n.

2/2/18. $ABCD$ is a trapezoid with $\overline{BC} \parallel \overline{AD}$, $\angle ADC = 57^\circ$, $\angle DAB = 33^\circ$, $BC = 6$, and $AD = 10$. M and N are the midpoints of $\overline{BC}$ and $\overline{AD}$, respectively.



(a) Find $\angle MNA$.

(b) Find MN.

3/2/18. The expression $\lfloor x \rfloor$ means the greatest integer that is smaller than or equal to x, and $\lceil x \rceil$ means the smallest integer that is greater than or equal to x. These functions are called the *floor function* and *ceiling function*, respectively. Find, with proof, a polynomial $f(n)$ equivalent to

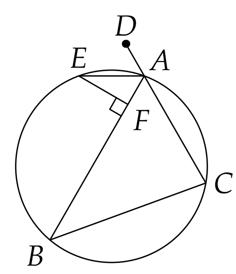
$$\sum_{k=1}^{n^2} \left(\left\lfloor \sqrt{k} \right\rfloor + \left\lceil \sqrt{k} \right\rceil \right)$$

for all positive integers n.

4/2/18. For every integer $k \geq 2$, find a formula (in terms of k) for the smallest positive integer n that has the following property:

> No matter how the elements of $\{1, 2, \ldots, n\}$ are colored red and blue, we can find k elements $a_1, a_2, \ldots, a_k$, where the a_i are not necessarily distinct elements, and an element b such that:
>
> (a) $a_1 + a_2 + \cdots + a_k = b$, and
>
> (b) all of the a_i's and b are the same color.

5/2/18. In triangle ABC, $AB = 8$, $BC = 7$, and $AC = 5$. We extend $\overline{AC}$ past A and mark point D on the extension, as shown. The bisector of $\angle DAB$ meets the circumcircle of $\triangle ABC$ again at E, as shown. We draw a line through E perpendicular to $\overline{AB}$. This line meets $\overline{AB}$ at point F. Find the length of $\overline{AF}$.



Round 2 Solutions must be submitted by **November 20, 2006**.

Please visit **http://www.usamts.org** for details about solution submission.