USA Mathematical Talent Search

Round 4 Problems

Year 17 — Academic Year 2005–2006

www.usamts.org

Please follow the rules below to ensure that your paper is graded properly.

1. If you have not already sent an Entry Form, download an Entry Form from the Forms page at

   `http://www.usamts.org/MyUSAMTS/U_MyForms.php`

   and submit the completed form with your solutions.

2. If you have already sent in an Entry Form and a Permission Form, you do not need to resend them.

3. Put your name and USAMTS ID# on **every page you submit**.

4. Once you send in your solutions, that submission is final. You cannot resubmit solutions.

5. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging into the site, then clicking on My USAMTS on the sidebar, then click Profile. If you are registered for the USAMTS and haven't received any email from us about the USAMTS, your email address is probably wrong in your Profile.

6. Do not fax solutions written in pencil.

7. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page.

8. In April, Round 4 results will be posted at www.usamts.org. To see your results, log in to the USAMTS page, then go to My USAMTS. Check that your email address in your USAMTS Profile is correct; you will receive an email when the scores are available.

9. Submit your solutions by March 13, 2006 (postmark deadline), via one (and only one!) of the methods below.

   (a) Email: solutions@usamts.org. Please see usamts.org for a list of acceptable file types. Do not send .doc Microsoft Word files.

   (b) Fax: (619) 445-2379 (Please include a cover sheet indicating the number of pages you are faxing, your name, and your User ID.)

   (c) Snail mail: USAMTS, P.O. Box 2090, Alpine, CA 91903–2090.

10. Re–read Items 1–9.

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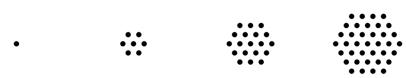
1/4/17. $\overline{AB}$ is a diameter of circle $\mathcal{C}_1$. Point P is on $\mathcal{C}_1$ such that $AP > BP$. Circle $\mathcal{C}_2$ is centered at P with radius PB. The extension of $\overline{AP}$ past P meets $\mathcal{C}_2$ at Q. Circle $\mathcal{C}_3$ is centered at A and is externally tangent to $\mathcal{C}_2$. R is on $\mathcal{C}_3$ such that $\overline{AR} \perp \overline{AQ}$. Circle $\mathcal{C}_4$ passes through A, Q, and R. Find, with proof, the ratio between the area of $\mathcal{C}_4$ and the area of $\mathcal{C}_1$, and show that this ratio is the same for all points P on $\mathcal{C}_1$ such that $AP > BP$.

2/4/17. *Centered hexagonal numbers* are the numbers of dots used to create hexagonal arrays of dots. The first four centered hexagonal numbers are 1, 7, 19, and 37, as shown below.



1 7 19 37

Centered Hexagonal Numbers

Consider an arithmetic sequence $1, a, b$ and a geometric sequence $1, c, d$, where a, b, c, and d are all positive integers and $a + b = c + d$. Prove that each centered hexagonal number is a possible value of a, and prove that each possible value of a is a centered hexagonal number.

3/4/17. We play a game. The pot starts at \$0. On every turn, you flip a fair coin. If you flip heads, I add \$100 to the pot. If you flip tails, I take all of the money out of the pot, and you are assessed a "strike." You can stop the game before any flip and collect the contents of the pot, but if you get 3 strikes, the game is over and you win nothing. Find, with proof, the expected value of your winnings if you follow an optimal strategy.

4/4/17. Find, with proof, all irrational numbers x such that both $x^3 - 6x$ and $x^4 - 8x^2$ are rational.

5/4/17. Sphere $\mathcal{S}$ is inscribed in cone $\mathcal{C}$. The height of $\mathcal{C}$ equals its radius, and both equal $12 + 12\sqrt{2}$. Let the vertex of the cone be A and the center of the sphere be B. Plane $\mathcal{P}$ is tangent to $\mathcal{S}$ and intersects segment $\overline{AB}$. X is the point on the intersection of $\mathcal{P}$ and $\mathcal{C}$ closest to A. Given that $AX = 6$, find the area of the region of $\mathcal{P}$ enclosed by the intersection of $\mathcal{C}$ and $\mathcal{P}$.

Round 4 Solutions must be submitted by **March 13, 2006**.

Please visit **http://www.usamts.org** for details about solution submission.