USA Mathematical Talent Search

Round 3 Problems

Year 17 — Academic Year 2005–2006

www.usamts.org

Please follow the rules below to ensure that your paper is graded properly.

1. If you have not already sent an Entry Form, download an Entry Form from the Forms page at

   `http://www.usamts.org/MyUSAMTS/U_MyForms.php`

   and submit the completed form with your solutions.

2. If you have already sent in an Entry Form and a Permission Form, you do not need to resend them.

3. Put your name and USAMTS ID# on **every page you submit**.

4. Once you send in your solutions, that submission is final. You cannot resubmit solutions.

5. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging into the site, then clicking on My USAMTS on the sidebar, then click Profile. If you are registered for the USAMTS and haven't received any email from us about the USAMTS, your email address is probably wrong in your Profile.

6. Do not fax solutions written in pencil.

7. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page.

8. In early March, Round 3 results will be posted at www.usamts.org. To see your results, log in to the USAMTS page, then go to My USAMTS. Check that your email address in your USAMTS Profile is correct; you will receive an email when the scores are available.

9. Submit your solutions by January 9, 2006 (postmark deadline), via one (and only one!) of the methods below.

   (a) Email: solutions@usamts.org. Please see usamts.org for a list of acceptable file types. Do not send .doc Microsoft Word files.

   (b) Fax: (619) 445-2379 (Please include a cover sheet indicating the number of pages you are faxing, your name, and your User ID.)

   (c) Snail mail: USAMTS, P.O. Box 2090, Alpine, CA 91903–2090.

10. Re–read Items 1–9.

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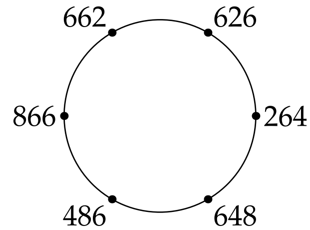
1/3/17.

For a given positive integer n, we wish to construct a circle of six numbers as shown at right so that the circle has the following properties:



(a) The six numbers are different three-digit numbers, none of whose digits is a 0.

(b) Going around the circle clockwise, the first two digits of each number are the last two digits, in the same order, of the previous number.

(c) All six numbers are divisible by n.

The example above shows a successful circle for $n = 2$. For each of $n = 3, 4, 5, 6, 7, 8, 9$, either construct a circle that satisfies these properties, or prove that it is impossible to do so.

2/3/17. Anna writes a sequence of integers starting with the number 12. Each subsequent integer she writes is chosen randomly with equal chance from among the positive divisors of the previous integer (including the possibility of the integer itself). She keeps writing integers until she writes the integer 1 for the first time, and then she stops. One such sequence is

$$12, 6, 6, 3, 3, 3, 1.$$

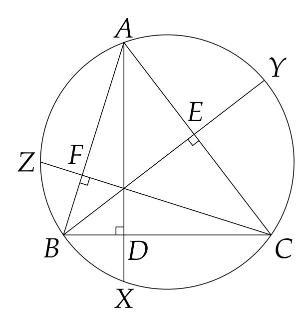
What is the expected value of the number of terms in Anna's sequence?

3/3/17.

Points A, B, and C are on a circle such that $\triangle ABC$ is an acute triangle. X, Y, and Z are on the circle such that AX is perpendicular to BC at D, BY is perpendicular to AC at E, and CZ is perpendicular to AB at F. Find the value of

$$\frac{AX}{AD} + \frac{BY}{BE} + \frac{CZ}{CF},$$

and prove that this value is the same for all possible A, B, C on the circle such that $\triangle ABC$ is acute.



4/3/17. Find, with proof, all triples of real numbers (a, b, c) such that all four roots of the polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + b$ are positive integers. (The four roots need not be distinct.)

Problem 5 on next page.

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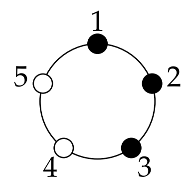
5/3/17. Lisa and Bart are playing a game. A round table has n lights evenly spaced around its circumference. Some of the lights are on and some of them off; the initial configuration is random. Lisa wins if she can get all of the lights turned on; Bart wins if he can prevent this from happening.

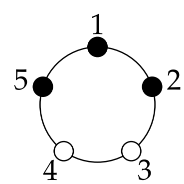
On each turn, Lisa chooses the positions at which to flip the lights, but *before* the lights are flipped, Bart, *knowing Lisa's choices*, can rotate the table to any position that he chooses (or he can leave the table as is). Then the lights in the positions that Lisa chose are flipped: those that are off are turned on and those that are on are turned off.

Here is an example turn for $n = 5$ (a white circle indicates a light that is on, and a black circle indicates a light that is off):

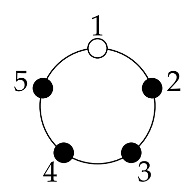
Initial Position.



Lisa says "1, 3, 4."
Bart rotates the table one position counterclockwise.



Lights in positions 1, 3, 4 are flipped.



Lisa can take as many turns as she needs to win, or she can give up if it becomes clear to her that Bart can prevent her from winning.

(a) Show that if $n = 7$ and initially at least one light is on and at least one light is off, then Bart can always prevent Lisa from winning.

(b) Show that if $n = 8$, then Lisa can always win in at most 8 turns.

Round 3 Solutions must be submitted by **January 9, 2006**.
Please visit **http://www.usamts.org** for details about solution submission.