USA Mathematical Talent Search

Round 2 Problems

Year 17 — Academic Year 2005–2006

www.usamts.org

Please follow the rules below to ensure that your paper is graded properly.

1. If you have not already sent an Entry Form, download an Entry Form from the Forms page at

   `http://www.usamts.org/MyUSAMTS/U_MyForms.php`

   and submit the completed form with your solutions.

2. If you have already sent in an Entry Form and a Permission Form, you do not need to resend them.

3. Put your name and USAMTS ID# on **every page you submit**.

4. Once you send in your solutions, that submission is final. You cannot resubmit solutions.

5. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging into the site, then clicking on My USAMTS on the sidebar, then click Profile. If you are registered for the USAMTS and haven't received any email from us about the USAMTS, your email address is probably wrong in your Profile.

6. Do not fax solutions written in pencil.

7. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page.

8. By the end of December, Round 2 results will be posted at www.usamts.org. To see your results, log in to the USAMTS page, then go to My USAMTS. Check that your email address in your USAMTS Profile is correct; you will receive an email when the scores are available.

9. Submit your solutions by November 21, 2005 (postmark deadline), via one (and only one!) of the methods below.

   (a) Email: solutions@usamts.org. Please see usamts.org for a list of acceptable file types. Do not send .doc Microsoft Word files.

   (b) Fax: (619) 445-2379 (Please include a cover sheet indicating the number of pages you are faxing, your name, and your User ID.)

   (c) Snail mail: USAMTS, P.O. Box 2090, Alpine, CA 91903–2090.

10. Re–read Items 1–9.

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1/2/17.

At the right is shown a 4×4 grid. We wish to fill in the grid such that each row, each column, and each 2×2 square outlined by the thick lines contains the digits 1 through 4. The first row has already been filled in. Find, with proof, the number of ways we can complete the rest of the grid.

1	2	3	4

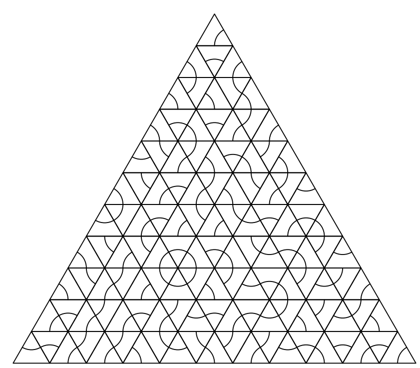
2/2/17. Write the number

$$\frac{1}{\sqrt{2} - \sqrt[3]{2}}$$

as the sum of terms of the form 2^q, where q is rational. (For example, $2^1 + 2^{-1/3} + 2^{8/5}$ is a sum of this form.) Prove that your sum equals $1/(\sqrt{2} - \sqrt[3]{2})$.

3/2/17.

An equilateral triangle is tiled with n^2 smaller congruent equilateral triangles such that there are n smaller triangles along each of the sides of the original triangle. The case $n = 11$ is shown at right. For each of the small equilateral triangles, we randomly choose a vertex V of the triangle and draw an arc with that vertex as center connecting the midpoints of the two sides of the small triangle with V as an endpoint. Find, with proof, the expected value of the number of full circles formed, in terms of n.



4/2/17. A teacher plays the game "Duck-Goose-Goose" with his class. The game is played as follows: All the students stand in a circle and the teacher walks around the circle. As he passes each student, he taps the student on the head and declares her a 'duck' or a 'goose'. Any student named a 'goose' leaves the circle immediately. Starting with the first student, the teacher tags students in the pattern: duck, goose, goose, duck, goose, goose, etc., and continues around the circle (re-tagging some former ducks as geese) until only one student remains. This remaining student is the winner.

For instance, if there are 8 students, the game proceeds as follows: student 1 (duck), student 2 (goose), student 3 (goose), student 4 (duck), student 5 (goose), student 6 (goose), student 7 (duck), student 8 (goose), student 1 (goose), student 4 (duck), student 7 (goose) and student 4 is the winner. Find, with proof, all values of n with $n > 2$ such that if the circle starts with n students, then the n^{th} student is the winner.

5/2/17. Given acute triangle $\triangle ABC$ in plane $\mathcal{P}$, a point Q in space is defined such that $\angle AQB = \angle BQC = \angle CQA = 90°$. Point X is the point in plane $\mathcal{P}$ such that $\overline{QX}$ is perpendicular to plane $\mathcal{P}$. Given $\angle ABC = 40°$ and $\angle ACB = 75°$, find $\angle AXC$.