USA Mathematical Talent Search

Round 2 Problems

Year 16 — Academic Year 2004–2005

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Please follow the rules below to ensure that your paper is graded properly.

1. Put your name, username, and USAMTS ID# on **every page you submit**.
2. Once you send in your solutions, that submission is final. You cannot resubmit solutions.
3. Confirm that your email address in your Art of Problem Solving Forum Profile is correct. You can do so by logging into the Forum, then clicking the Profile button near the top of the Forum. If you are registered for the USAMTS and haven't received any email from us about the USAMTS, your email address is probably wrong in your Forum Profile.
4. Do not fax solutions written in pencil.
5. No single page should contain solutions to more than one problem.
6. By the end of October, Round 1 results will be posted at www.usamts.org. To see your results, log in to the Art of Problem Solving Forum, then go to My USAMTS in the USAMTS pages.
7. Submit your solutions by November 22, 2004 (postmark deadline), via one of the methods below.
   (a) Email: solutions@usamts.org. Please see usamts.org for a list of acceptable file types. Do not send .doc Microsoft Word files.
   (b) Fax: (619) 445-2379
   (c) Snail mail: USAMTS, P.O. Box 2090, Alpine, CA 91903–2090.
8. Re–read item 1.

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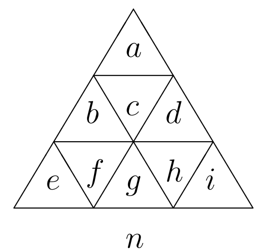
1/2/16. The numbers 1 through 9 can be arranged in the triangles labeled a through i illustrated on the right so that the numbers in each of the 2×2 triangles sum to the same value n; that is

$$a + b + c + d = b + e + f + g = d + g + h + i = n.$$

For each possible sum n, show such an arrangement, labeled with the sum as shown at right. Prove that there are no possible arrangements for any other values of n.

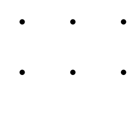


2/2/16. Call a number $a - b\sqrt{2}$ with a and b both positive integers *tiny* if it is closer to zero than any number $c - d\sqrt{2}$ such that c and d are positive integers with $c < a$ and $d < b$. Three numbers which are tiny are $1 - \sqrt{2}$, $3 - 2\sqrt{2}$, and $7 - 5\sqrt{2}$. Without using a calculator or computer, prove whether or not each of the following is tiny:

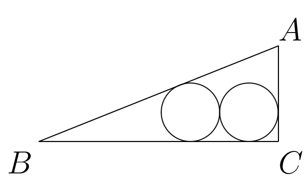
$$(a)\ 58 - 41\sqrt{2}, \qquad\qquad (b)\ 99 - 70\sqrt{2}.$$

3/2/16. A set is *reciprocally whole* if its elements are distinct integers greater than 1 and the sum of the reciprocals of all those elements is exactly 1. Find a set S, as small as possible, that contains two reciprocally whole subsets, I and J, which are distinct but not necessarily disjoint (meaning they may share elements, but they may not be the same subset). Prove that no set with fewer elements than S can contain two reciprocally whole subsets.

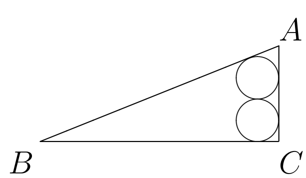
4/2/16. How many quadrilaterals in the plane have four of the nine points $(0,0)$, $(0,1)$, $(0,2)$, $(1,0)$, $(1,1)$, $(1,2)$, $(2,0)$, $(2,1)$, $(2,2)$ as vertices? Do count both concave and convex quadrilaterals, but do not count figures where two sides cross each other or where a vertex angle is 180°. Rigorously verify that no quadrilateral was skipped or counted more than once.



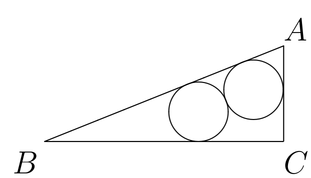
5/2/16. Two circles of equal radius can tightly fit inside right triangle ABC, which has $AB = 13$, $BC = 12$, and $CA = 5$, in the three positions illustrated below. Determine the radii of the circles in each case.



Case (i)



Case (ii)



Case (iii)

Round 2 Solutions must be submitted by **November 22, 2004**.

Please visit **http://www.usamts.org** for details about solution submission.