USA Mathematical Talent Search

Round 1 Problems

Year 34 — Academic Year 2022-2023

www.usamts.org

Important information:

1. **You must show your work and prove your answers on all problems.** If you just send a numerical answer with no proof for a problem other than Problem 1, you will get no more than 1 point.

2. Put your name, username, and USAMTS ID# on **every page you submit**.

3. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page.

4. Submit your solutions by **October 11, 2022** via one (and only one!) of the methods below:

   (a) Web: Log on to www.usamts.org to upload a PDF file containing your solutions. (No other file type will be accepted.)
   Deadline: 10 PM Eastern / 7 PM Pacific on October 11, 2022.

   (b) Mail: USAMTS
   55 Exchange Place
   Suite 603
   New York, NY 10005
   Deadline: Solutions must be postmarked on or before October 11, 2022.

5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.

6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging on to www.usamts.org and visiting the "My USAMTS" pages.

7. Round 1 results will be posted at www.usamts.org when available. To see your results, log on to the USAMTS website, then go to "My USAMTS". You will also receive an email when your scores and comments are available (provided that you did item #6 above).

These are only part of the complete rules.
Please read the entire rules at www.usamts.org.

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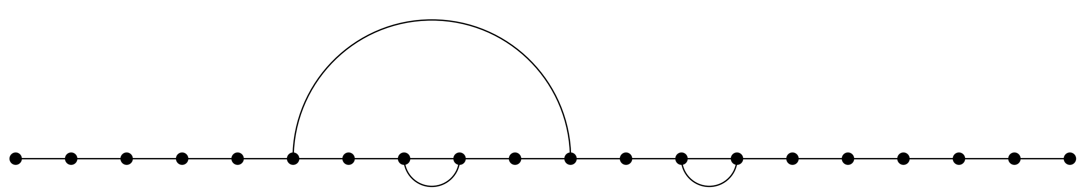
Each problem is worth 5 points.

1/1/34. Shown is a segment of length 19, marked with 20 points dividing the segment into 19 segments of length 1. Draw 20 semicircular arcs, each of whose endpoints are two of the 20 marked points, satisfying all of the following conditions:

1. When the drawing is complete, there will be:
    - 8 arcs with diameter 1,
    - 6 arcs with diameter 3,
    - 4 arcs with diameter 5,
    - 2 arcs with diameter 7.
2. Each marked point is the endpoint of exactly two arcs: one above the segment and one below the segment.
3. No two distinct arcs can intersect except at their endpoints.
4. No two distinct arcs can connect the same pair of points. (That is, there can be no full circles.)

Three arcs have already been drawn for you.



There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the conditions of the problem. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

2/1/34. Given a sphere, a *great circle* of the sphere is a circle on the sphere whose diameter is also a diameter of the sphere. For a given positive integer n, the surface of a sphere is divided into several regions by n great circles, and each region is colored black or white. We say that a coloring is *good* if any two adjacent regions (that share an arc as boundary, not just a finite number of points) have different colors. Find, with proof, all positive integers n such that in every good coloring with n great circles, the sum of the areas of the black regions is equal to the sum of the areas of the white regions.

3/1/34. Prove that there is a unique 1000-digit number N in base 2022 with the following properties:

1. All of the digits of N (in base 2022) are 1's or 2's, and
2. N is a multiple of the base-10 number 2^{1000}.

(Note that you must prove both that such a number exists and that there is not more than one such number. You do not have to write down the number! In fact, please don't!)

4/1/34. Grogg and Winnie are playing a game using a deck of 50 cards numbered 1 through 50. They take turns with Grogg going first. On each turn a player chooses a card from the deck—this choice is made deliberately, *not* at random—and then adds it to one of two piles (both piles are empty at the start of the game). After all 50 cards are in the two piles, the values of the cards in each pile are summed, and Winnie wins the positive difference of the sums of the two piles, in dollars. (For instance, if the first pile has cards summing to 510 and the second pile has cards summing to 765, then Winnie wins $255.) Winnie wants to win as much as possible, and Grogg wants Winnie to win as little as possible. If they both play with perfect strategy, find (with proof) the amount that Winnie wins.

5/1/34. We call a positive integer n *sixish* if $n = p(p+6)$, where p and $p+6$ are prime numbers. For example, $187 = 11 \cdot 17$ is sixish, but $475 = 19 \cdot 25$ is not sixish.

Define a function f on positive integers such that $f(n)$ is the sum of the squares of the positive divisors of n. For example, $f(10) = 1^2 + 2^2 + 5^2 + 10^2 = 130$.

(a) Find, with proof, an irreducible polynomial function $g(x)$ with integer coefficients such that $f(n) = g(n)$ for all sixish n. ("Irreducible" means that $g(x)$ cannot be factored as the product of two polynomials of smaller degree with integer coefficients.)

(b) We call a positive integer n *pseudo-sixish* if n is not sixish but nonetheless $f(n) = g(n)$, where $g(n)$ is the polynomial function that you found in part (a). Find, with proof, all pseudo-sixish positive integers.

Problems by USAMTS Staff.
Round 1 Solutions must be submitted by **October 11, 2022**.
Please visit **http://www.usamts.org** for details about solution submission.