# USA Mathematical Talent Search

Round 2 Problems

Year 33 — Academic Year 2021-2022

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**Important information:**

1. **You must show your work and prove your answers on all problems.** If you just send a numerical answer with no proof for a problem other than Problem 1, you will get no more than 1 point.

2. Put your name, username, and USAMTS ID# on **every page you submit**.

3. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page.

4. Submit your solutions by **November 29, 2021** via one (and only one!) of the methods below:

   (a) Web: Log on to `www.usamts.org` to upload a PDF file containing your solutions. (No other file type will be accepted.)
   **Deadline: 10 PM Eastern / 7 PM Pacific on November 29, 2021.**

   (b) Mail: USAMTS
   55 Exchange Place
   Suite 603
   New York, NY 10005
   **Deadline: Solutions must be postmarked on or before November 29, 2021.**

5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.

6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging on to `www.usamts.org` and visiting the "My USAMTS" pages.

7. Round 2 results will be posted at `www.usamts.org` when available. To see your results, log on to the USAMTS website, then go to "My USAMTS". You will also receive an email when your scores and comments are available (provided that you did item #6 above).

**These are only part of the complete rules.**
**Please read the entire rules on `www.usamts.org`.**

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**Each problem is worth 5 points.**

**1/2/33.** A $5 \times 5$ **Latin Square** is a $5 \times 5$ grid of squares in which each square contains one of the numbers 1 through 5 such that every number appears exactly once in each row and column. A partially completed grid (with numbers in some of the squares) is **puzzle-ready** if there is a unique way to fill in the remaining squares to complete a Latin Square.

Below is a partially completed grid with seven squares filled in and an additional three squares shaded. Determine what numbers must be filled into the shaded squares to make the grid (now with ten squares filled in) puzzle-ready, and then complete the Latin Square.

There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

| | | | | |
|---|---|---|---|---|
| 1 | | | | |
| | 1 | 3 | | |
| 2 | 3 | | | 5 |
| | | | 5 | |
| | | | | |

**2/2/33.** Let $n$ be a fixed positive integer. Which is greater?

1. The number of $n$-tuples of integers whose largest value is 7 and whose smallest value is 0; or

2. The number of ordered triples $(A, B, C)$ that satisfy the following property: $A, B, C$ are subsets of $\{1, 2, 3, \ldots, n\}$, and neither $C \subseteq A \cup B$, nor $B \subseteq A \cup C$.

Your answer can be: (1), (2), the two counts are equal, or it depends on $n$.

**3/2/33.** Let $x$ and $y$ be distinct real numbers such that

$$\sqrt{x^2+1} + \sqrt{y^2+1} = 2021x + 2021y.$$

Find, with proof, the value of

$$(x + \sqrt{x^2+1})(y + \sqrt{y^2+1}).$$

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**4/2/33.** Let $ABC$ be a scalene triangle, and let $X$, $Y$, $Z$ be points on sides $BC$, $CA$, $AB$, respectively. Let $I$ and $O$ denote the incenter and circumcenter, respectively, of triangle $ABC$. Suppose that

$$\frac{BX - CX}{BA - CA} = \frac{CY - AY}{CB - AB} = \frac{AZ - BZ}{AC - BC}.$$

Prove that there exists a point $P$ on line $IO$ such that $\overline{PX} \perp \overline{BC}$, $\overline{PY} \perp \overline{CA}$, and $\overline{PZ} \perp \overline{AB}$.

**5/2/33.** For a finite nonempty set $A$ of positive integers, $A = \{a_1, a_2, \dots, a_n\}$, we say the **calamitous complement** of $A$ is the set of all positive integers $k$ for which there do ***not*** exist nonnegative integers $w_1, w_2, \dots, w_n$ with

$$k = a_1 w_1 + a_2 w_2 + \cdots + a_n w_n.$$

The calamitous complement of $A$ is denoted $cc(A)$. For example,

$$cc(\{5, 6, 9\}) = \{1, 2, 3, 4, 7, 8, 13\}.$$

Find all pairs of positive integers $a, b$ with $1 < a < b$ for which there exists a set $G$ satisfying all of the following properties:

1. $G$ is a set of at most three positive integers,
2. $cc(\{a, b\})$ and $cc(G)$ are both finite sets, and
3. $cc(G) = cc(\{a, b\}) \cup \{m\}$ for some $m$ not in $cc(\{a, b\})$.

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*Problems by Ryan Kuroyama, Michael Tang, Evan Chen, and USAMTS Staff.*
Round 2 Solutions must be submitted by **November 29, 2021**.
Please visit **http://www.usamts.org** for details about solution submission.