USA Mathematical Talent Search

Round 2 Problems

Year 31 — Academic Year 2019-2020

www.usamts.org

Important information:

1. **You must show your work and prove your answers on all problems.** If you just send a numerical answer with no proof for a problem other than Problem 1, you will get no more than 1 point.
2. Put your name, username, and USAMTS ID# on **every page you submit**.
3. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page.
4. Submit your solutions by **December 2, 2019** via one (and only one!) of the methods below:
   (a) Web: Log on to www.usamts.org to upload a PDF file containing your solutions. (No other file type will be accepted.)
   Deadline: 8 PM Eastern / 5 PM Pacific on December 2, 2019.
   (b) Mail: USAMTS
   55 Exchange Place
   Suite 603
   New York, NY 10005
   Deadline: Solutions must be postmarked on or before December 2, 2019.
5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.
6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging onto www.usamts.org and visiting the "My USAMTS" pages.
7. Round 2 results will be posted at www.usamts.org when available. To see your results, log on to the USAMTS website, then go to "My USAMTS". You will also receive an email when your scores and comments are available (provided that you did item #6 above).

These are only part of the complete rules.
Please read the entire rules on www.usamts.org.

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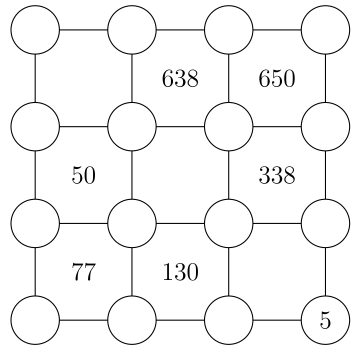
Each problem is worth 5 points.

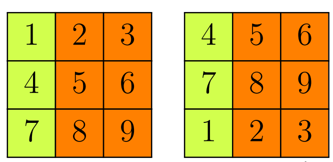
1/2/31. Fill in each empty white circle with a number from 1 to 16 so that each number is used exactly once. One number has been given to you. If a square has a given number inside and its four vertices contain the numbers a, b, c, d in clockwise order, then the number inside the square must be equal to $(a+c)(b+d)$.



There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

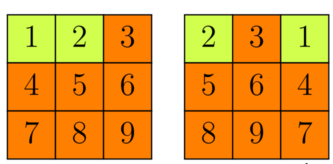
2/2/31. A 3×3 grid of blocks is labeled from 1 through 9. Cindy paints each block orange or lime with equal probability and gives the grid to her friend Sophia.

Sophia then plays with the grid of blocks. She can take the top row of blocks and move it to the bottom, as shown.



Grid A Grid A'

She can also take the leftmost column of blocks and move it to the right end, as shown.



Grid B Grid B'

Sophia calls the grid of blocks *citrus* if it is impossible for her to use a sequence of the moves described above to obtain another grid with the same coloring but a different numbering scheme. For example, Grid B is *citrus*, but Grid A is not *citrus* because moving the top row of blocks to the bottom results in a grid with a different numbering but the same coloring as Grid A.

What is the probability that Sophia receives a *citrus* grid of blocks?

3/2/31. Call a quadruple of positive integers (a, b, c, d) *fruitful* if there are infinitely many integers m such that $\gcd(am + b, cm + d) = 2019$. Find all possible values of $|ad - bc|$ over fruitful quadruples (a, b, c, d).

4/2/31. Princess Pear has 100 jesters with heights $1, 2, \ldots, 100$ inches. On day n with $1 \leq n \leq 100$, Princess Pear holds a court with all her jesters with height at most n inches, and she receives two candied cherries from every group of 6 jesters with a median height of $n - 50$ inches. A jester can be part of multiple groups.

On day 101, Princess Pear summons all 100 jesters to court one final time. Every group of 6 jesters with a median height of 50.5 inches presents one more candied cherry to the Princess. How many candied cherries does Princess Pear receive in total?

Please provide a numerical answer (with justification).

5/2/31. Let ABC be a triangle with circumcenter O, A-excenter I_A, B-excenter I_B, and C-excenter I_C. The incircle of $\triangle ABC$ is tangent to sides $\overline{BC}, \overline{CA}$, and $\overline{AB}$ at D, E, and F respectively. Lines $\overline{I_B E}$ and $\overline{I_C F}$ intersect at P. If the line through O perpendicular to $\overline{OP}$ passes through I_A, prove that $\angle A = 60^\circ$.

An excenter is the point of concurrency among one internal angle bisector and two external angle bisectors of a triangle.