USA Mathematical Talent Search

Round 1 Problems

Year 31 — Academic Year 2019-2020

www.usamts.org

Important information:

1. **You must show your work and prove your answers on all problems.** If you just send a numerical answer with no proof for a problem other than Problem 1, you will get no more than 1 point.

2. Put your name and USAMTS ID# on **every page you submit**.

3. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page.

4. Submit your solutions by **October 15, 2019** via one (and only one!) of the methods below:

   (a) Web: Log on to www.usamts.org to upload a PDF file containing your solutions. (No other file type will be accepted.)
   Deadline: 8 PM Eastern / 5 PM Pacific on October 15, 2019.

   (b) Mail: USAMTS
   55 Exchange Place
   Suite 603
   New York, NY 10005
   Deadline: Solutions must be postmarked on or before October 15, 2019.

5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.

6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging on to www.usamts.org and visiting the "My USAMTS" pages.

7. Round 1 results will be posted at www.usamts.org when available. To see your results, log on to the USAMTS website, then go to "My USAMTS". You will also receive an email when your scores and comments are available (provided that you did item #6 above).

These are only part of the complete rules.
Please read the entire rules on www.usamts.org.

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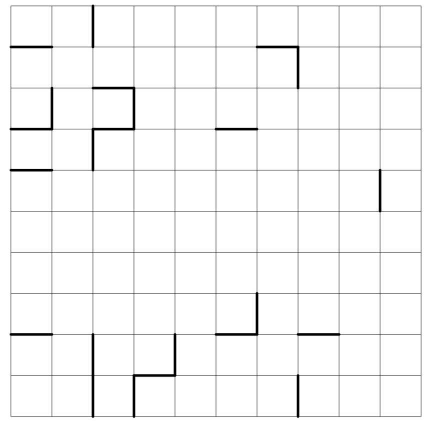
Each problem is worth 5 points.

1/1/31. Partition the grid into 1 by 1 squares and 1 by 2 dominoes in either orientation, marking dominoes with a line connecting the two adjacent squares, and 1 by 1 squares with an asterisk ($*$). No two 1 by 1 squares can share a side. A *border* is a grid segment between two adjacent squares that contain dominoes of opposite orientations. All borders have been marked with thick lines in the grid.



There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

2/1/31. Let x, y, and z be real numbers greater than 1. Prove that if $x^y = y^z = z^x$, then $x = y = z$.

3/1/31. Circle ω is inscribed in unit square $PLUM$, and points I and E lie on ω such that U, I, and E are collinear. Find, with proof, the greatest possible area for $\triangle PIE$.

4/1/31. A group of 100 friends stands in a circle. Initially, one person has 2019 mangos, and no one else has mangos. The friends split the mangos according to the following rules:

- *sharing*: to share, a friend passes two mangos to the left and one mango to the right.
- *eating*: the mangos must also be eaten and enjoyed. However, no friend wants to be selfish and eat too many mangos. Every time a person eats a mango, they must also pass another mango to the right.

A person may only *share* if they have at least three mangos, and they may only *eat* if they have at least two mangos. The friends continue sharing and eating, until so many mangos have been eaten that no one is able to share or eat anymore.

Show that there are exactly eight people stuck with mangos, which can no longer be shared or eaten.

5/1/31. Let n be a positive integer. For integers a, b with $0 \le a, b \le n-1$, let $r_n(a, b)$ denote the remainder when ab is divided by n. If S_n denotes the sum of all n^2 remainders $r_n(a, b)$, prove that

$$\frac{1}{2} - \frac{1}{\sqrt{n}} \le \frac{S_n}{n^3} \le \frac{1}{2}.$$

Problems by Michael Tang, Billy Swartworth, and USAMTS Staff.

Round 1 Solutions must be submitted by **October 15, 2019**.

Please visit **http://www.usamts.org** for details about solution submission.