USA Mathematical Talent Search

Round 3 Problems

Year 30 — Academic Year 2018-2019

www.usamts.org

Important information:

1. **You must show your work and prove your answers on all problems.** If you just send a numerical answer with no proof for a problem other than Problem 1, you will get no more than 1 point.

2. Put your name and USAMTS ID# on **every page you submit**.

3. No single page should contain solutions to more than one problem; every solution you submit should begin on a new page.

4. Submit your solutions by January 2, 2019, via one (and only one!) of the methods below:

   (a) Web: Log on to www.usamts.org to upload a PDF file containing your solutions. (No other file type will be accepted.)
   Deadline: 8 PM Eastern / 5 PM Pacific on January 2, 2019

   (b) Mail: USAMTS
   55 Exchange Place
   Suite 603
   New York, NY 10005
   (Solutions must be postmarked on or before January 2.)

5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.

6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging onto www.usamts.org and visiting the “My USAMTS” pages.

7. Round 3 results will be posted at www.usamts.org when available. To see your results, log on to the USAMTS website, then go to “My USAMTS”. You will also receive an email when your scores and comments are available (provided that you did item #6 above).

These are only part of the complete rules.
Please read the entire rules on www.usamts.org.

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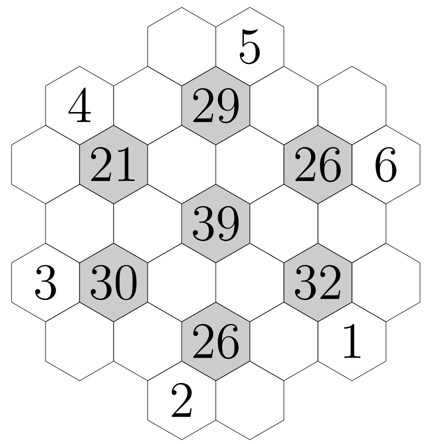
Each problem is worth 5 points.

1/3/30. Fill in each white hexagon with a positive digit from 1 to 9. Some digits have been given to you. Each of the seven gray hexagons touches six hexagons; these six hexagons must contain six distinct digits, and the sum of these six digits must equal the number inside the gray hexagon.

You do not need to prove that your answer is the only one possible; you merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)



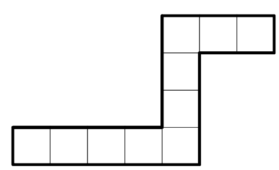
2/3/30. Lizzie writes a list of fractions as follows. First, she writes $\frac{1}{1}$, the only fraction whose numerator and denominator add to 2. Then she writes the two fractions whose numerator and denominator add to 3, in increasing order of denominator. Then she writes the three fractions whose numerator and denominator sum to 4 in increasing order of denominator. She continues in this way until she has written all the fractions whose numerator and denominator sum to at most 1000. So Lizzie's list looks like:

$$\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{3}{1}, \frac{2}{2}, \frac{1}{3}, \frac{4}{1}, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, \ldots, \frac{1}{999}.$$

Let p_k be the product of the first k fractions in Lizzie's list. Find, with proof, the value of $p_1 + p_2 + \cdots + p_{499500}$.

3/3/30. Cyclic quadrilateral $ABCD$ has $AC \perp BD$, $AB + CD = 12$, and $BC + AD = 13$. Find the greatest possible area for $ABCD$.

4/3/30. An *eel* is a polyomino formed by a path of unit squares that makes two turns in opposite directions. (Note that this means the smallest eel has four cells.) For example, the polyomino shown at right is an eel. What is the maximum area of a 1000×1000 grid of unit squares that can be covered by eels without overlap?



5/3/30. The sequence $\{a_n\}$ is defined by $a_0 = 1$, $a_1 = 2$, and for $n \geq 2$,

$$a_n = a_{n-1}^2 + (a_0 a_1 \cdots a_{n-2})^2 .$$

Let k be a positive integer, and let p be a prime factor of a_k. Show that $p > 4(k-1)$.

Problems by David Altizio, Nikolai Beluhov, Billy Swartworth, Michael Tang, and USAMTS Staff.

Round 3 Solutions must be submitted by **January 2, 2019**.

Please visit **http://www.usamts.org** for details about solution submission.