USA Mathematical Talent Search

Round 3 Problems

Year 29 — Academic Year 2017-2018

www.usamts.org

Important information:

1. **You must show your work and prove your answers on all problems.** If you just send a numerical answer with no proof for a problem other than Problem 1, you will get no more than 1 point.

2. Put your name and USAMTS ID# on **every page you submit**.

3. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page, and you should only submit work on one side of each piece of paper.

4. Submit your solutions by January 2, 2018, via one (and only one!) of the methods below:

   (a) Web: Log on to www.usamts.org to upload a PDF file containing your solutions. (No other file type will be accepted.)
   Deadline: 8 PM Eastern / 5 PM Pacific on January 2, 2018

   (b) Mail: USAMTS
   P.O. Box 4499
   New York, NY 10163
   (Solutions must be postmarked on or before January 2, 2018.)

5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.

6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging onto www.usamts.org and visiting the "My USAMTS" pages.

7. Round 3 results will be posted at www.usamts.org when available. To see your results, log on to the USAMTS website, then go to "My USAMTS". You will also receive an email when your scores and comments are available (provided that you did item #6 above).

These are only part of the complete rules.
Please read the entire rules on www.usamts.org.

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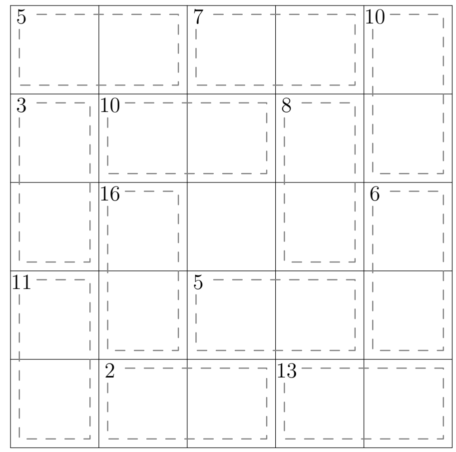
Each problem is worth 5 points.

1/3/29. Fill in each cell of the grid with a positive digit so that the following conditions hold:



1. each row and column contains five distinct digits;
2. for any cage containing multiple cells of a row, the label on the cage is the GCD of the sum of the digits in the cage and the sum of the digits in the whole row; and
3. for any cage containing multiple cells of a column, the label on the cage is the GCD of the sum of the digits in the cage and the sum of the digits in the whole column.

You do not need to prove that your answer is the only one possible; you merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

2/3/29. Let q be a real number. Suppose there are three distinct positive integers a, b, c such that $q + a, q + b, q + c$ is a geometric progression. Show that q is rational.

3/3/29. Let ABC be an equilateral triangle with side length 1. Let A_1 and A_2 be the trisection points of AB with A_1 closer to A, B_1 and B_2 be the trisection points of BC with B_1 closer to B, and C_1 and C_2 be the trisection points of CA with C_1 closer to C. Grogg has an orange equilateral triangle the size of triangle $A_1B_1C_1$. He puts the orange triangle over triangle $A_1B_1C_1$ and then rotates it about its center in the shortest direction until its vertices are over $A_2B_2C_2$. Find the area of the region that the orange triangle traveled over during its rotation.

4/3/29. A positive integer is called *uphill* if the digits in its decimal representation form an increasing sequence from left to right. That is, a number $\overline{a_1a_2\cdots a_n}$ is uphill if $a_i \leq a_{i+1}$ for all i. For example, 123 and 114 are both uphill. Suppose a polynomial $P(x)$ with rational coefficients takes on an integer value for each uphill positive integer x. Is it necessarily true that $P(x)$ takes on an integer value for each integer x?

5/3/29. Let n be a positive integer. Aavid has a card deck consisting of $2n$ cards, each colored with one of n colors such that every color is on exactly two of the cards. The $2n$ cards are randomly ordered in a stack. Every second, he removes the top card from the stack and places the card into an area called the pit. If the other card of that color also happens to be in the pit, Aavid collects both cards of that color and discards them from the pit.

Of the $(2n)!$ possible original orderings of the deck, determine how many have the following property: at every point, the pit contains cards of at most two distinct colors.