USA Mathematical Talent Search

Round 2 Problems

Year 28 — Academic Year 2016–2017

www.usamts.org

Important information:

1. **You must show your work and prove your answers for all problems.** If you just send a numerical answer with no proof for a problem other than Problem 1, you will get no more than 1 point.
2. Put your name and USAMTS ID# on **every page you submit**.
3. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page.
4. Submit your solutions by November 28, 2016, via one (and only one!) of the methods below:
   (a) Web: Log on to `www.usamts.org` to upload a PDF file containing your solutions. (No other file type will be accepted.)
   Deadline: 8 PM Eastern / 5 PM Pacific on November 28, 2016
   (b) Mail: USAMTS
   PO Box 4499
   New York, NY 10163
   (Solutions must be postmarked on or before November 28, 2016.)
5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.
6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging onto `www.usamts.org` and visiting the "My USAMTS" pages.
7. Round 2 results will be posted at `www.usamts.org` when available. To see your results, log on to the USAMTS website, then go to "My USAMTS". You will also receive an email when your scores and comments are available (provided that you did item #6 above).

These are only part of the complete rules.
Please read the entire rules on `www.usamts.org`.

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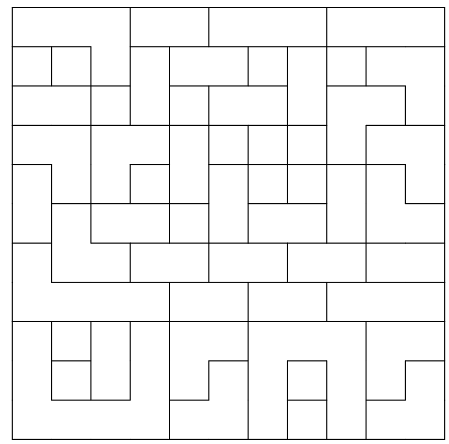
Each problem is worth 5 points.

1/2/28. Shade in some of the regions in the grid to the right so that the shaded area is equal for each of the 11 rows and columns. Regions must be fully shaded or fully unshaded, at least one region must be shaded, and the area of shaded regions must be at most half of the whole grid.



You do not need to prove that your answer is the only one possible; you merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

2/2/28. Find all triples of three-digit positive integers $x < y < z$ with x, y, z in arithmetic progression and $x, y, z + 1000$ in geometric progression.

For this problem, you may use calculators or computers to gain an intuition about how to solve the problem. However, your final submission should include mathematical derivations or proofs and should not be a solution by exhaustive search.

3/2/28. Suppose m and n are relatively prime positive integers. A regular m-gon and a regular n-gon are inscribed in a circle. Let d be the minimum distance **in degrees** (of the arc along the circle) between a vertex of the m-gon and a vertex of the n-gon. What is the maximum possible value of d?

4/2/28. On Binary Island, residents communicate using special paper. Each sheet of paper is a $1 \times n$ row of initially uncolored squares. To send a message, each square on the paper must be colored either red or green. Unfortunately the paper on the island has become damaged, and each sheet of paper has 10 random consecutive squares each of which is randomly colored red or green.

Malmer and Weven would like to develop a scheme that allows them to send messages of length 2016 between one another. They would like to be able to send any message of length 2016, and they want their scheme to work with perfect accuracy. What is the smallest value of n for which they can develop such a strategy?

Note that when creating a message, one can see which 10 squares are randomly colored and what colors they are. One also knows on which square the message begins, and on which square the message ends.

5/2/28. Let $n \geq 4$ and $y_1, \ldots, y_n$ real with

$$\sum_{k=1}^{n} y_k = \sum_{k=1}^{n} k y_k = \sum_{k=1}^{n} k^2 y_k = 0$$

and

$$y_{k+3} - 3y_{k+2} + 3y_{k+1} - y_k = 0$$

for $1 \leq k \leq n-3$. Prove that

$$\sum_{k=1}^{n} k^3 y_k = 0.$$