USA Mathematical Talent Search

Round 3 Problems

Year 27 — Academic Year 2015–2016

www.usamts.org

Important information:

1. **You must prove your answers on all problems** (other than Problem 1). This means showing all necessary steps to demonstrate that your result is true; however, please leave out any work that is unnecessary to the proof. If you just send a numerical answer with no proof for a problem other than Problem 1, you will get no more than 1 point.

2. Put your name and USAMTS ID# on **every page you submit**.

3. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page, and you should only submit work on one side of each piece of paper.

4. Submit your solutions by January 4, 2016, via one (and only one!) of the methods below:

   (a) Web: Log on to www.usamts.org to upload a PDF file containing your solutions. (No other file type will be accepted.)
   Deadline: 3 PM Eastern / Noon Pacific on January 4, 2016

   (b) Mail: USAMTS
   P.O. Box 4499
   New York, NY 10163
   (Solutions must be postmarked on or before January 4, 2016. We **strongly recommend** that you keep a copy of your solutions and that you pay for tracking on the mailing. With large envelopes, there have been significant delays in the past.)

5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.

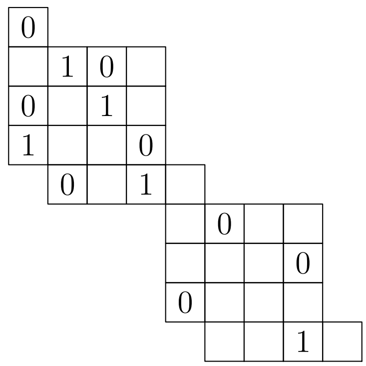
6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging onto www.usamts.org and visiting the "My USAMTS" pages.

7. Round 3 results will be posted at www.usamts.org when available. To see your results, log on to the USAMTS website, then go to "My USAMTS". You will also receive an email when your scores and comments are available (provided that you did item #6 above).

These are only part of the complete rules.
Please read the entire rules on www.usamts.org.

Each problem is worth 5 points.

1/3/27. Fill in each space of the grid with either a 0 or a 1 so that all 16 strings of four consecutive numbers across and down are distinct.



You do not need to prove that your answer is the only one possible; you merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

2/3/27. Fames is playing a computer game with falling two-dimensional blocks. The playing field is 7 units wide and infinitely tall with a bottom border. Initially the entire field is empty. Each turn, the computer gives Fames a 1×3 solid rectangular piece of three unit squares. Fames must decide whether to orient the piece horizontally or vertically and which column(s) the piece should occupy (3 consecutive columns for horizontal pieces, 1 column for vertical pieces). Once he confirms his choice, the piece is dropped straight down into the playing field in the selected columns, stopping all three of the piece's squares as soon as the piece hits either the bottom of the playing field or any square from another piece. All of the pieces must be contained completely inside the playing field after dropping and cannot partially occupy columns.

If at any time a row of 7 spaces is all filled with squares, Fames scores a point.

Unfortunately, Fames is playing in *invisible mode*, which prevents him from seeing the state of the playing field or how many points he has, and he has already arbitrarily dropped some number of pieces without remembering what he did with them or how many there were.

For partial credit, find a strategy that will allow Fames to eventually earn at least one more point. For full credit, find a strategy for which Fames can correctly announce "I have earned at least one more point" and know that he is correct.

3/3/27. For $n > 1$, let a_n be the number of zeroes that $n!$ ends with when written in base n. Find the maximum value of $\frac{a_n}{n}$.

4/3/27. Let $\triangle ABC$ be a triangle with $AB < AC$. Let the angle bisector of $\angle BAC$ meet BC at D, and let M be the midpoint of $\overline{BC}$. Let P be the foot of the perpendicular from B to $\overline{AD}$. Extend $\overline{BP}$ to meet $\overline{AM}$ at Q. Show that $\overline{DQ}$ is parallel to $\overline{AB}$.

5/3/27. Let $a_1, a_2, \ldots, a_{100}$ be a sequence of integers. Initially, $a_1 = 1, a_2 = -1$ and the remaining numbers are 0. After every second, we perform the following process on the sequence: for $i = 1, 2, \ldots, 99$, replace a_i with $a_i + a_{i+1}$, and replace a_{100} with $a_{100} + a_1$. (All of this is done simultaneously, so each new term is the sum of two terms of the sequence from before any replacements.) Show that for any integer M, there is some index i and some time t for which $|a_i| > M$ at time t.