USA Mathematical Talent Search

Round 2 Problems

Year 27 — Academic Year 2015–2016

www.usamts.org

Important information:

1. **You must prove your answers on all problems** (other than Problem 1). This means showing all necessary steps to demonstrate that your result is true; however, please leave out any work that is unnecessary to the proof. If you just send a numerical answer with no proof for a problem other than Problem 1, you will get no more than 1 point.

2. Put your name and USAMTS ID# on **every page you submit**.

3. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page, and you should only submit work on one side of each piece of paper.

4. Submit your solutions by November 30, 2015, via one (and only one!) of the methods below:

   (a) Web: Log on to `www.usamts.org` to upload a PDF file containing your solutions. (No other file type will be accepted.)
   Deadline: 3 PM Eastern / Noon Pacific on November 30

   (b) Mail: USAMTS
   P.O. Box 4499
   New York, NY 10163
   (Solutions must be postmarked on or before November 30. We **strongly recommend** that you keep a copy of your solutions and that you pay for tracking on the mailing. With large envelopes, there have been significant delays in the past.)

5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.

6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging onto `www.usamts.org` and visiting the "My USAMTS" pages.

7. Round 1 results will be posted at `www.usamts.org` when available. To see your results, log on to the USAMTS website, then go to "My USAMTS". You will also receive an email when your scores and comments are available (provided that you did item #6 above).

These are only part of the complete rules.
Please read the entire rules on `www.usamts.org`.

USA Mathematical Talent Search

Round 2 Problems

Year 27 — Academic Year 2015–2016

www.usamts.org

Each problem is worth 5 points.

1/2/27. In the grid to the right, the shortest path through unit squares between the pair of 2's has length 2. Fill in some of the unit squares in the grid so that

(i) exactly half of the squares in each row and column contain a number,

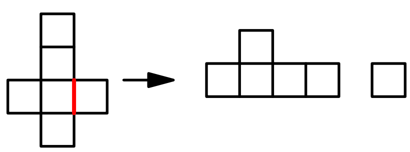
(ii) each of the numbers 1 through 12 appears exactly twice, and

(iii) for $n = 1, 2, \ldots, 12$, the shortest path between the pair of n's has length exactly n.

				6			
	10	1					
			2				8
	5			2			
					7	9	
				3			

You do not need to prove that your answer is the only one possible; you merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

2/2/27. A net for a polyhedron is cut along an edge to give two **pieces**. For example, we may cut a cube net along the red edge to form two pieces as shown.



Are there two distinct polyhedra for which this process may result in the same two pairs of pieces? If you think the answer is no, prove that no pair of polyhedra can result in the same two pairs of pieces. If you think the answer is yes, prove an example; a clear example will suffice as proof.

3/2/27. For all positive integers n, show that

$$\frac{1}{n} \sum_{k=1}^{n} \frac{k \cdot k! \cdot \binom{n}{k}}{n^k} = 1.$$

4/2/27. Find all polynomials $P(x)$ with integer coefficients such that, for all integers a and b, $P(a+b) - P(b)$ is a multiple of $P(a)$.

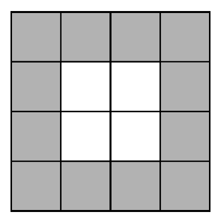
USA Mathematical Talent Search
Round 2 Problems
Year 27 — Academic Year 2015–2016
www.usamts.org

5/2/27. Let $n > 1$ be an even positive integer. A $2n \times 2n$ grid of unit squares is given, and it is partitioned into n^2 contiguous 2×2 blocks of unit squares. A subset S of the unit squares satisfies the following properties:

(i) For any pair of squares A, B in S, there is a sequence of squares in S that starts with A, ends with B, and has any two consecutive elements sharing a side; and

(ii) In each of the 2×2 blocks of squares, at least one of the four squares is in S.

An example for $n = 2$ is shown below, with the squares of S shaded and the four 2×2 blocks of squares outlined in bold.



In terms of n, what is the minimum possible number of elements in S?