USA Mathematical Talent Search

Round 1 Problems

Year 27 — Academic Year 2015–2016

www.usamts.org

Important information:

1. **You must show your work and prove your answers on all problems.** If you just send a numerical answer with no proof for a problem other than Problem 1, you will get no more than 1 point.
2. Put your name and USAMTS ID# on **every page you submit**.
3. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page, and you should only submit work on one side of each piece of paper.
4. Submit your solutions by October 19, 2015, via one (and only one!) of the methods below:
   (a) Web: Log on to www.usamts.org to upload a PDF file containing your solutions. (No other file type will be accepted.)
   Deadline: 3 PM Eastern / Noon Pacific on October 19
   (b) Mail: USAMTS
   P.O. Box 4499
   New York, NY 10163
   (Solutions must be postmarked on or before October 19. We **strongly recommend** that you keep a copy of your solutions and that you pay for tracking on the mailing. With large envelopes, there have been significant delays in the past.)
5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.
6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging onto www.usamts.org and visiting the "My USAMTS" pages.
7. Round 1 results will be posted at www.usamts.org when available. To see your results, log on to the USAMTS website, then go to "My USAMTS". You will also receive an email when your scores and comments are available (provided that you did item #6 above).

These are only part of the complete rules.
Please read the entire rules on www.usamts.org.

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Each problem is worth 5 points.

1/1/27. Fill in the spaces of the grid to the right with positive integers so that in each 2×2 square with top left number a, top right number b, bottom left number c, and bottom right number d, either $a + d = b + c$ or $ad = bc$.

3	9			
	11		7	2
10				16
15				
20	36			32

You do not need to prove that your answer is the only one possible; you merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

2/1/27. Suppose a, b, and c are distinct positive real numbers such that

$$abc = 1000,$$
$$bc(1 - a) + a(b + c) = 110.$$

If $a < 1$, show that $10 < c < 100$.

3/1/27. Let P be a convex n-gon in the plane with vertices labeled $V_1, \dots, V_n$ in counterclockwise order. A point Q not outside P is called a *balancing point* of P if, when the triangles $QV_1V_2, QV_2V_3, \dots, QV_{n-1}V_n, QV_nV_1$ are alternately colored blue and green, the total areas of the blue and green regions are the same. Suppose P has exactly one balancing point. Show that the balancing point must be a vertex of P.

4/1/27. Several players try out for the USAMTS basketball team, and they all have integer heights and weights when measured in centimeters and pounds, respectively. In addition, they all weigh less in pounds than they are tall in centimeters. All of the players weigh at least 190 pounds and are at most 197 centimeters tall, and there is exactly one player with every possible height-weight combination.

The USAMTS wants to field a competitive team, so there are some strict requirements.

(i) If person P is on the team, then anyone who is at least as tall and at most as heavy as P must also be on the team.

(ii) If person P is on the team, then no one whose weight is the same as P's height can also be on the team.

Assuming the USAMTS team can have any number of members (including zero), how many different basketball teams can be constructed?

5/1/27. Find all positive integers n that have distinct positive divisors $d_1, d_2, \ldots, d_k$, where $k > 1$, that are in arithmetic progression and

$$n = d_1 + d_2 + \cdots + d_k.$$

Note that $d_1, d_2, \ldots, d_k$ do not have to be all the divisors of n.