USA Mathematical Talent Search

Round 3 Problems

Year 24 — Academic Year 2012–2013

www.usamts.org

Important information:

1. **You must show your work and prove your answers on all problems.** If you just send a numerical answer with no proof for a problem other than Problem 1, you will get no more than 1 point.

2. Put your name and USAMTS ID# on **every page you submit**.

3. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page, and you should only submit work on one side of each piece of paper.

4. Submit your solutions by January 7, 2013, via one (and only one!) of the methods below:

   (a) Web: Log on to www.usamts.org to upload a PDF file containing your solutions. (No other file type will be accepted.)
   Deadline: 3 PM Eastern / Noon Pacific on January 7, 2013

   (b) Mail: USAMTS, P.O. Box 2090, Alpine, CA 91903–2090.
   (Solutions must be postmarked on or before January 7.)

5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.

6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging onto www.usamts.org and visiting the "My USAMTS" pages.

7. Round 3 results will be posted at www.usamts.org when available. To see your results, log on to the USAMTS website, then go to "My USAMTS". You will also receive an email when your scores and comments are available (provided that you did item #6 above).

These are only part of the complete rules.
Please read the entire rules on www.usamts.org.

USA Mathematical Talent Search

Round 3 Problems

Year 24 — Academic Year 2012–2013

www.usamts.org

Each problem is worth 5 points.

1/3/24. **(Corrected from an earlier release.)** In the 8×8 grid shown, fill in 12 of the grid cells with the numbers 1–12 so that the following conditions are satisfied:

1. Each cell contains at most one number, and each number from 1–12 is used exactly once.
2. Two cells that both contain numbers may not touch, even at a point.
3. A clue outside the grid pointing at a row or column gives the sum of all of the numbers in that row or column. Rows and columns without clues have an unknown sum.[1]

	24 ▼	1 ▼	3 ▼		20 ▼	13 ▼		11 ▼
3 ▶								
18 ▶								
21 ▶								
20 ▶								
13 ▶								
3 ▶								

You do not need to prove that your configuration is the only one possible; you merely need to find a configuration that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

2/3/24. Palmer and James work at a dice factory, placing dots on dice. Palmer builds his dice correctly, placing the dots so that 1, 2, 3, 4, 5, and 6 dots are on separate faces. In a fit of mischief, James places his 21 dots on a die in a peculiar order, putting some nonnegative integer number of dots on each face, but not necessarily in the correct configuration. Regardless of the configuration of dots, both dice are unweighted and have equal probability of showing each face after being rolled.

Then Palmer and James play a game. Palmer rolls one of his normal dice and James rolls his peculiar die. If they tie, they roll again. Otherwise the person with the larger roll is the winner. What is the maximum probability that James wins? Give one example of a peculiar die that attains this maximum probability.

3/3/24. In quadrilateral $ABCD$, $\angle DAB = \angle ABC = 110^\circ$, $\angle BCD = 35^\circ$, $\angle CDA = 105^\circ$, and AC bisects $\angle DAB$. Find $\angle ABD$.

4/3/24. **(Corrected from an earlier release.)** Denote by $\lfloor x \rfloor$ the greatest integer[2] less than or equal to x. Let $m \geq 2$ be an integer, and let s be a real number between 0 and 1. Define an infinite sequence of real numbers $a_1, a_2, a_3, \ldots$ by setting $a_1 = s$ and $a_k = m a_{k-1} - (m-1)\lfloor a_{k-1} \rfloor$ for all $k \geq 2$. For example, if $m = 3$ and $s = \frac{5}{8}$, then we get $a_1 = \frac{5}{8}$, $a_2 = \frac{15}{8}$, $a_3 = \frac{29}{8}$, $a_4 = \frac{39}{8}$, and so on.

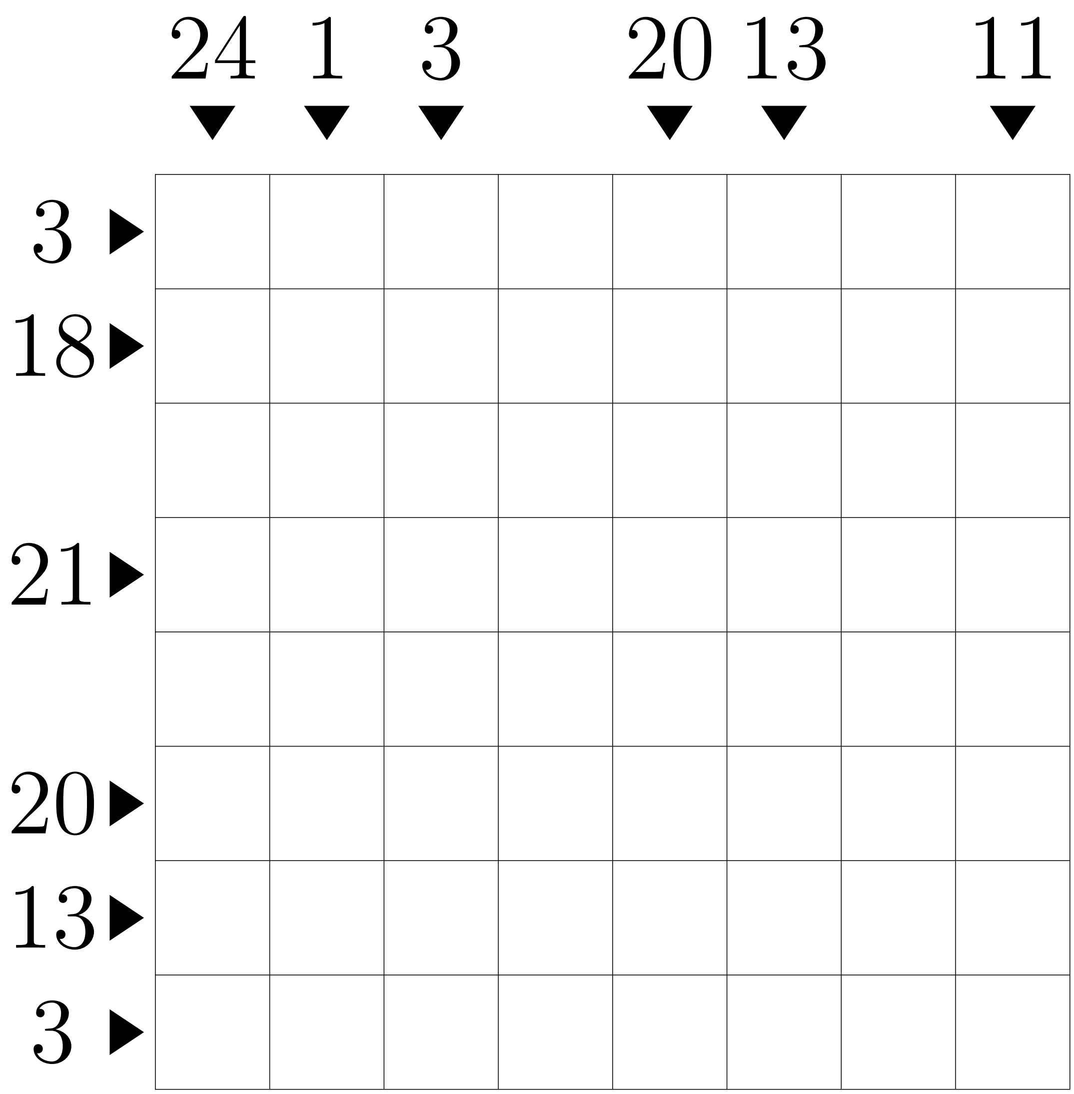
Call the sequence $a_1, a_2, a_3, \ldots$ **orderly** if we can find rational numbers b, c such that $\lfloor a_n \rfloor = \lfloor bn + c \rfloor$ for all $n \geq 1$. With the example above where $m = 3$ and $s = \frac{5}{8}$, we get an orderly sequence since $\lfloor a_n \rfloor = \left\lfloor \frac{3n}{2} - \frac{3}{2} \right\rfloor$ for all n. Show that if s is an irrational number and $m \geq 2$ is any integer, then the sequence $a_1, a_2, a_3, \ldots$ is **not** an orderly sequence.

5/3/24. Let P and Q be two polynomials with real coefficients such that P has degree greater than 1 and

$$P(Q(x)) = P(P(x)) + P(x).$$

Show that $P(-x) = P(x) + x$.

Larger diagram for Problem 1/3/24.



Notes

[1]The second sentence about rows and columns without clues was not in the original version. It was added to make the problem clearer.

[2]The original version erroneously said "greatest positive integer less than or equal to x."