USA Mathematical Talent Search

Round 1 Problems

Year 24 — Academic Year 2012–2013

`www.usamts.org`

Important information:

1. **You must show your work and prove your answers on all problems.** If you just send a numerical answer with no proof for a problem other than Problem 3, you will get no more than 1 point.
2. Put your name and USAMTS ID# on **every page you submit**.
3. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page, and you should only submit work on one side of each piece of paper.
4. Submit your solutions by Monday, October 22, 2012, via one (and only one!) of the methods below:
   (a) Web: Log on to `www.usamts.org` to upload a PDF file containing your solutions. (No other file type will be accepted.)
   Deadline: 3 PM Eastern / Noon Pacific on October 22
   (b) Mail: USAMTS, P.O. Box 2090, Alpine, CA 91903–2090.
   (Solutions must be postmarked on or before October 22.)
5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.
6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging onto `www.usamts.org` and visiting the "My USAMTS" pages.
7. Round 1 results will be posted at `www.usamts.org` when available. To see your results, log on to the USAMTS website, then go to "My USAMTS". You will also receive an email when your scores and comments are available (provided that you did item #6 above).

These are only part of the complete rules.
Please read the entire rules on `www.usamts.org`.

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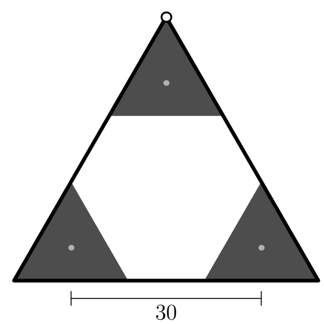
Each problem is worth 5 points.

1/1/24. Several children were playing in the ugly tree when suddenly they all fell.

- Roger hit branches A, B, and C in that order on the way down.
- Sue hit branches D, E, and F in that order on the way down.
- Gillian hit branches G, A, and C in that order on the way down.
- Marcellus hit branches B, D, and H in that order on the way down.
- Juan-Phillipe hit branches I, C, and E in that order on the way down.

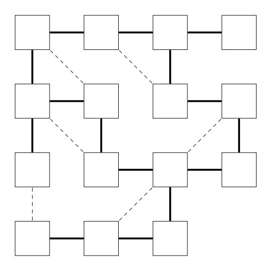
Poor Mikey hit every branch A through I on the way down. Given only this information, in how many different orders could he have hit these 9 branches on the way down?

2/1/24. Three wooden equilateral triangles of side length 18 inches are placed on axles as shown in the diagram to the right. Each axle is 30 inches from the other two axles. A 144-inch leather band is wrapped around the wooden triangles, and a dot at the top corner is painted as shown. The three triangles are then rotated at the same speed and the band rotates without slipping or stretching. Compute the length of the path that the dot travels before it returns to its initial position at the top corner.



3/1/24. The **symmetric difference**, $\triangle$, of a pair of sets is the set of elements in exactly one set. For example,

$$\{1,2,3\}\triangle\{2,3,4\} = \{1,4\}.$$



There are fifteen nonempty subsets of $\{1,2,3,4\}$. Assign each subset to exactly one of the squares in the grid to the right so that the following conditions are satisfied.

(i) If A and B are in squares connected by a solid line then $A\triangle B$ has exactly one element.

(ii) If A and B are in squares connected by a dashed line then the largest element of A is equal to the largest element of B.

You do not need to prove that your configuration is the only one possible; you merely need to find a configuration that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

4/1/24. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. Let m be a positive integer, $m \geq 3$. For every integer i with $1 \leq i \leq m$, let

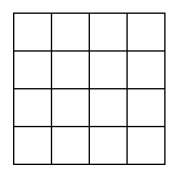
$$S_{m,i} = \left\{ \left\lfloor \frac{2^m - 1}{2^{i-1}} n - 2^{m-i} + 1 \right\rfloor : n = 1, 2, 3, \dots \right\}.$$

For example, for $m = 3$,

$$\begin{aligned} S_{3,1} &= \{\lfloor 7n - 3 \rfloor : n = 1, 2, 3, \dots\} \\ &= \{4, 11, 18, \dots\}, \\ S_{3,2} &= \left\{ \left\lfloor \frac{7}{2} n - 1 \right\rfloor : n = 1, 2, 3, \dots \right\} \\ &= \{2, 6, 9, \dots\}, \\ S_{3,3} &= \left\{ \left\lfloor \frac{7}{4} n \right\rfloor : n = 1, 2, 3, \dots \right\} \\ &= \{1, 3, 5, \dots\}. \end{aligned}$$

Prove that for all $m \geq 3$, each positive integer occurs in exactly one of the sets $S_{m,i}$.

5/1/24. An ordered quadruple (y_1, y_2, y_3, y_4) is **quadratic** if there exist real numbers a, b, and c such that



$$y_n = an^2 + bn + c$$

for $n = 1, 2, 3, 4$.

Prove that if 16 numbers are placed in a 4×4 grid such that all four rows are quadratic and the first three columns are also quadratic then the fourth column must also be quadratic.

(We say that a row is quadratic if its entries, in order, are quadratic. We say the same for a column.)

Larger diagram for Problem 3/1/24.