USA Mathematical Talent Search

Round 1 Problems

Year 23 — Academic Year 2011–2012

www.usamts.org

Important information:

1. **You must show your work and prove your answers on all problems.** If you just send a numerical answer for a problem with no proof, you will get no more than 1 point.

2. Put your name and USAMTS ID# on **every page you submit**.

3. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page, and you should only submit work on one side of each piece of paper.

4. Submit your solutions by Tuesday, October 11, 2011, via one (and only one!) of the methods below:

   (a) Web: Log on to www.usamts.org to upload a PDF file containing your solutions. (No other file type will be accepted.)
   Deadline: 3 PM Eastern / Noon Pacific on October 11

   (b) Mail: USAMTS, P.O. Box 2090, Alpine, CA 91903–2090.
   (Solutions must be postmarked on or before October 11.)

5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.

6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging onto www.usamts.org and visiting the "My USAMTS" pages.

7. Round 1 results will be posted at www.usamts.org when available. To see your results, log on to the USAMTS website, then go to "My USAMTS". You will also receive an email when your scores and comments are available (provided that you did item #6 above).

These are only part of the complete rules.
Please read the entire rules on www.usamts.org.

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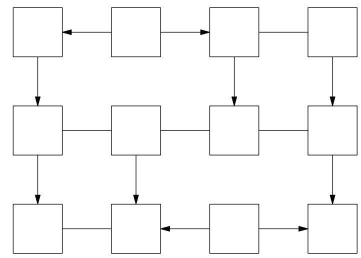
Each problem is worth 5 points.

1/1/23. The grid on the right has 12 boxes and 15 edges connecting boxes. In each box, place one of the six integers from 1 to 6 such that the following conditions hold:



- For each possible pair of distinct numbers from 1 to 6, there is exactly one edge connecting two boxes with that pair of numbers.
- If an edge has an arrow, then it points from a box with a smaller number to a box with a larger number.

You do not need to prove that your configuration is the only one possible; you merely need to find a configuration that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

2/1/23. Find all integers a, b, c, d, and e, such that

$$\begin{aligned}
a^2 &= a + b - 2c + 2d + e - 8,\\
b^2 &= -a - 2b - c + 2d + 2e - 6,\\
c^2 &= 3a + 2b + c + 2d + 2e - 31,\\
d^2 &= 2a + b + c + 2d + 2e - 2,\\
e^2 &= a + 2b + 3c + 2d + e - 8.
\end{aligned}$$

3/1/23. **(Corrected from an earlier release.)** You have 14 coins, dated 1901 through 1914. Seven of these coins are real and weigh 1.000 ounce each. The other seven are counterfeit and weigh 0.999 ounces each. You do not know which coins are real or counterfeit. You also cannot tell which coins are real by look or feel.

Fortunately for you, Zoltar the Fortune-Weighing Robot is capable of making very precise measurements. You may place any number of coins in each of Zoltar's two hands and Zoltar will do the following:

- If the weights in each hand are equal, Zoltar tells you so and returns all of the coins.
- If the weight in one hand is heavier than the weight in the other, then Zoltar takes one coin, at random, from the heavier hand as tribute. Then Zoltar tells you which hand was heavier, and returns the remaining coins to you.[1]

[1] In the earlier version, this sentence read, "Then Zoltar tells you *the result of the measurement*, and returns the remaining coins to you." The correction clarifies that "the result of the measurement" was meant to refer only to which hand was heavier, not to the actual weight in either hand.

Your objective is to identify a single real coin that Zoltar has not taken as tribute. Is there a strategy that guarantees this? If so, then describe the strategy and why it works. If not, then prove that no such strategy exists.

4/1/23. Let $ABCDEF$ and $ABC'D'E'F'$ be regular planar hexagons in three-dimensional space with side length 1, such that $\angle EAE' = 60°$. Let $\mathcal{P}$ be the convex polyhedron whose vertices are A, B, C, C', D, D', E, E', F, and F'.

(a) Find the radius r of the largest sphere that can be enclosed in polyhedron $\mathcal{P}$.

(b) Let $\mathcal{S}$ be a sphere enclosed in polyhedron $\mathcal{P}$ with radius r (as derived in part (a)). The set of possible centers of $\mathcal{S}$ is a line segment $\overline{XY}$. Find the length XY.

5/1/23. In the game of Tristack Solitaire, you start with three stacks of cards, each with a different positive integer number of cards. At any time, you can double the number of cards in any one stack of cards by moving cards from exactly one other, larger, stack of cards to the stack you double. You win the game when any two of the three stacks have the same number of cards.

For example, if you start with stacks of 3, 5, and 7 cards, then you have three possible legal moves:

- You may move 3 cards from the 5-card stack to the 3-card stack, leaving stacks of 6, 2, and 7 cards.
- You may move 3 cards from the 7-card stack to the 3-card stack, leaving stacks of 6, 5, and 4 cards.
- You may move 5 cards from the 7-card stack to the 5-card stack, leaving stacks of 3, 10, and 2 cards.

Can you win Tristack Solitaire from any starting position? If so, then give a strategy for winning. If not, then explain why.