USA Mathematical Talent Search

Round 2 Solutions

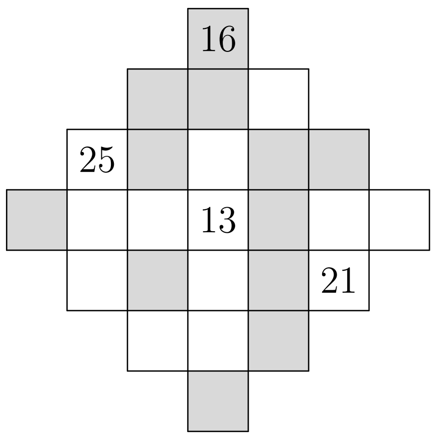
Year 34 — Academic Year 2022–2023

`www.usamts.org`

1/2/34. Fill in the grid below with the numbers 1 through 25, with each number used exactly once, subject to the following constraints:

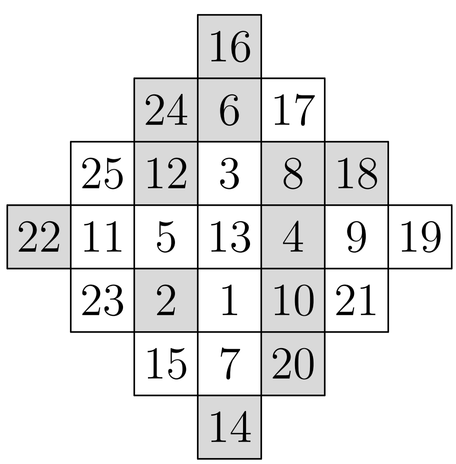
1. Each shaded square contains an even number, and each unshaded square contains an odd number.
2. For any pair of squares that share a side, if x and y are the two numbers in those squares, then either $x \geq 2y$ or $y \geq 2x$.

Four numbers have been filled in already.



There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the conditions of the problem. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

Solution



2/2/34. Grogg's favorite positive integer is $n \geq 2$, and Grogg has a lucky coin that comes up heads with some fixed probability p, where $0 < p < 1$. Once each day, Grogg flips his coin, and if it comes up heads, he does two things:

1. He eats a cookie.
2. He then flips the coin n more times. If the result of these n flips is $n-1$ heads and 1 tail (in any order), he eats another cookie.

He never eats a cookie except as a result of his coin flips. Find all possible values of n and p such that the expected value of the number of cookies that Grogg eats each day is exactly 1.

Solution

For (1), the probability that the coin comes up heads is p, so this event contributes an expected value of $p \cdot 1$ cookies.

For (2), binomial probability tells us that the probability that the n flips result in $n-1$ heads and 1 tail is $np^{n-1}(1-p)$. Since Grogg only does the n coin flips if the initial flip was heads, we multiply by p. The second event contributes an additional expected value of $pnp^{n-1}(1-p)$ cookies.

Thus, the expected number of cookies that Grogg eats each day is $p(1+np^{n-1}(1-p))$. Since we want to find all possible values of n and p that give us an expected value of 1, we solve the equation $p(1+np^{n-1}(1-p)) = 1$. The equation rearranges and factors as $(p-1)(np^n-1) = 0$. Since $p < 1$, we must have $np^n = 1$, so $\boxed{p = \dfrac{1}{\sqrt[n]{n}}}$ is the solution for any n.

3/2/34. Let $n \geq 3$ be a positive integer. Alex and Lizzie play a game. Alex chooses n positive integers (not necessarily distinct), writes them on a blackboard, and does nothing further. Then, Lizzie is allowed to pick some of the numbers—but not all of them—and replace them each by their average. For example, if $n = 7$ and the numbers Alex writes on the blackboard to start are $1, 2, 4, 5, 9, 4, 11$, then on her first turn Lizzie could pick $1, 4, 9$, erase them, and replace them each with the number $\frac{1+4+9}{3}$, leaving on the blackboard the numbers $\frac{14}{3}, 2, \frac{14}{3}, 5, \frac{14}{3}, 4, 11$. Lizzie can repeat this process of selecting and averaging some numbers as often as she wants. Lizzie wins the game if eventually all of the numbers written on the blackboard are equal. Find all positive integers $n \geq 3$ such that no matter what numbers Alex picks, Lizzie can win the game.

Solution

Lizzie wins if and only if n is composite.

Suppose n is composite, so that $n = rs$ for some $1 < r, s < n$. Lizzie can win in $r + s$ turns, as follows. First, Lizzie arranges the numbers into an $r \times s$ matrix. On her first r turns, Lizzie takes the s numbers in one of the r rows, in succession, and replaces them with their average. Note that after this, each column of the matrix is identical. Then on her next s terms, Lizzie takes the r numbers in one of the s columns, in succession, and replaces them with their average. Since each column was identical, the result will be that all of the numbers in the matrix will be equal.

Conversely, suppose that n is prime, and Alex writes $n - 1$ 1's and one 2 on the board. Then at the end of the game, each number must be $\frac{n+1}{n}$. However, since Lizzie must always choose $k < n$ numbers, she will always multiply any existing denominator of a number on the board by a factor of k when averaged. In particular, since n is prime there is no way to get a n in any denominator at any time during the game.

4/2/34. A *lattice point* of the coordinate plane is a point (x, y) in which both x and y are integers. Let $k \geq 2$ be a positive integer. Find the smallest positive integer c_k (which may depend on k) such that every lattice point can be colored with one of c_k colors, subject to the following two conditions:

1. If (x, y) and (a, b) are two distinct neighboring points; that is, $|x-a| \leq 1$ and $|y-b| \leq 1$, then (x, y) and (a, b) must be different colors.
2. If (x, y) and (a, b) are two lattice points such that $x \equiv a \pmod{k}$ and $y \equiv b \pmod{k}$, then (x, y) and (a, b) must be the same color.

Solution

The answer is $c_k = \begin{cases} 4 & \text{if } k \text{ is even,} \\ 9 & \text{if } k = 3, \\ 5 & \text{if } k > 3 \text{ is odd.} \end{cases}$

Call two distinct lattice points *adjacent* if they satisfy the condition in (1). Note that $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(1, 1)$ are all mutually adjacent, so regardless of k, we need $c_k \geq 4$. Also note that any two adjacent points will differ in parity in at least one coordinate.

If k is even, then 4 colors suffice: color all (odd, odd) points with one color, $(\text{odd}, \text{even})$ points with a second color, $(\text{even}, \text{odd})$ points with a third color, and $(\text{even}, \text{even})$ points with a fourth color. Then since adjacent points differ in parity in at least one coordinate, they will always have different colors, and furthermore since k is even, two points with each pair of coordinates equivalent modulo k will have each pair of coordinates that are respectively equal in parity, and thus be the same color.

If $k = 3$, then the color pattern must repeat in a 3×3 grid. This implies that each point in any 3×3 grid within the coordinate plane must have a different color. So $c_3 = 9$.

If $k > 3$ is odd, then 4 colors is insufficient: if we attempt to use only 4 colors, then as noted earlier $(0, 0)$, $(0, 1)$, $(1, 0)$, and $(1, 1)$ must use all four colors, and an easy induction shows that the points $(2n + 1, 0)$ and $(2n + 1, 1)$ must use the same two colors as $(1, 0)$ and $(1, 1)$ (in either order). But then $(k, 0)$ will not match $(0, 0)$, a contradiction.

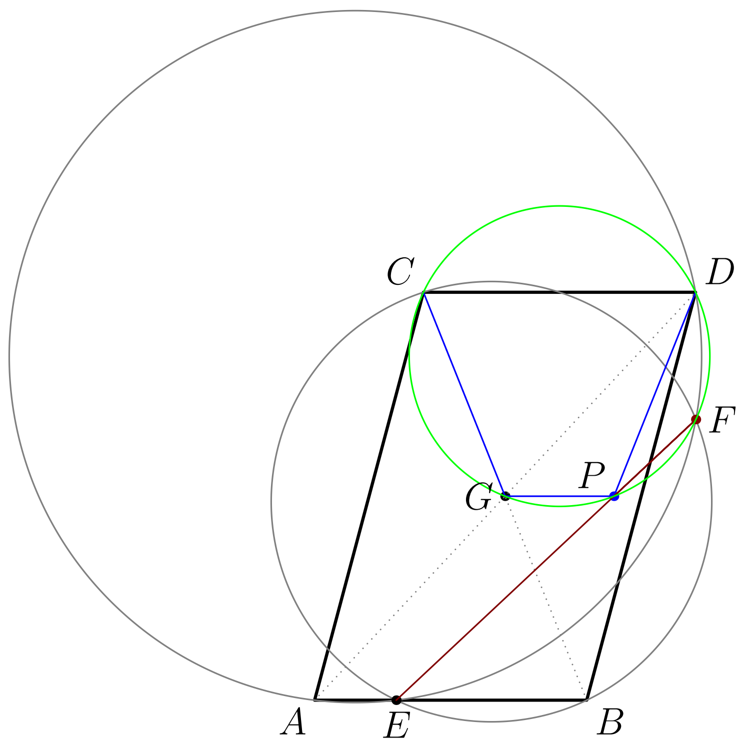
However, 5 colors is sufficient, as follows. Color the points (i, j) with $0 \leq i, j \leq 4$ using five colors A,B,C,D,E as below:

D	E	A	B	C
B	C	D	E	A
E	A	B	C	D
C	D	E	A	B
A	B	C	D	E

Then extend this coloring to all (i, j) with $0 \le i < k$ and $0 \le j \le 4$ inductively by coloring $(i+2, j)$ the same as (i, j). Finally, extend this coloring to all (i, j) with $0 \le i, j < k$ inductively by coloring $(i, j+2)$ the same as (i, j), and copy this $k \times k$ grid across the entire coordinate plane.

5/2/34. Let r and s be positive real numbers, and let $A = (0,0)$, $B = (1,0)$, $C = (r,s)$, and $D = (r+1,s)$ be points on the coordinate plane. Find a point P, whose coordinates are expressed in terms of r and s, with the following property: if E is any point on the interior of line segment $\overline{AB}$, and F is the unique point other than E that lies on the circumcircles of triangles BCE and ADE, then P lies on line $\overleftrightarrow{EF}$.

Solution



The answer is $P = \boxed{\left(\frac{3r+1}{2}, \frac{s}{2}\right)}$.

Note that $ABDC$ is a parallelogram (since $AB = CD = 1$ and $\overline{AB} \parallel \overline{CD}$), and let $G = \left(\frac{r+1}{2}, \frac{s}{2}\right)$ be the intersection of its diagonals $\overline{AD}$ and $\overline{BC}$.

Claim: C, D, F, G are the vertices of a cyclic quadrilateral.

Proof: Assume F is on the same side of $\overline{CD}$ as E, so that the quadrilateral is $CDFG$.

It suffices to show that $\angle DFC = \angle DGC$. We have

$$\begin{aligned}
\angle DFC = \angle DFE - \angle CFE &= 180^\circ - \angle DAE - \angle CFE \qquad \text{(because } AEFD \text{ is cyclic)}\\
&= 180^\circ - \angle DAE - \angle CBE \qquad \text{(because } CFBE \text{ is cyclic)}\\
&= \angle AGB \qquad \text{(using } \triangle ABG\text{)}\\
&= \angle DGC \qquad \text{(using vertical angles at } G\text{)}
\end{aligned}$$

(The case where F is on the opposite side of $\overline{CD}$ as E is a similar angle chase.) This proves the claim. □

Let P be the point of intersection of this circumcircle with $\overleftrightarrow{EF}$ (other that F itself).

Claim: $CDPG$ is an isosceles trapezoid.

Proof: It suffices to show that $\angle DGP = \angle CPG$. But this is true, again assuming that F is on the same side of $\overline{CD}$ as E, because

$$\begin{aligned}
\angle DGP &= 180^\circ - \angle DFP \qquad \text{(because } DFPG \text{ is cyclic)}\\
&= \angle DAE \qquad \text{(because } DFEA \text{ is cyclic)}\\
&= \angle CDG \qquad \text{(because } ABDC \text{ is a parallelogram)}\\
&= \angle CPG \qquad \text{(because } CDPG \text{ is cyclic)}
\end{aligned}$$

(The case where F is on the opposite side of $\overline{CD}$ as E is a similar angle chase.) □

Hence regardless of the choice of E, point P is determined on $\overleftrightarrow{EF}$ as the unique point such that $CDPG$ is a cyclic isosceles trapezoid. Chasing coordinates gives $P = \left(\frac{3r+1}{2}, \frac{s}{2}\right)$.