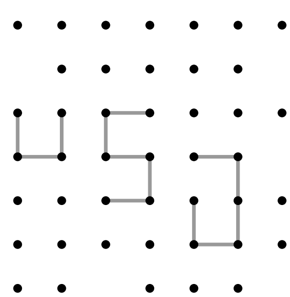
1/3/33. In the grid below, draw horizontal and vertical segments of unit length joining pairs of adjacent dots (some have been given to you) so that

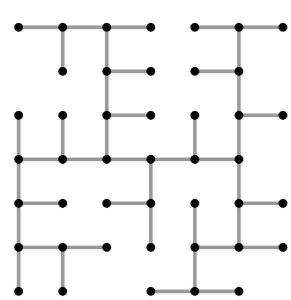
1. every dot is connected by line segments to exactly 1 or 3 adjacent dots,
2. any dot can be reached from any other dot by following a path of segments, and
3. no area is completely enclosed by segments.

Note: "Unit length" is the length between two adjacent dots when there is no missing dot between them. For example, we cannot draw a vertical line segment down from the dot in the top right corner because the length of this segment would be 2 units.

There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)



Solution



2/3/33. Sydney the squirrel is at $(0,0)$ and is trying to get to $(2021,2022)$. She can move only by reflecting her position over any line that can be formed by connecting two lattice points, provided that the reflection puts her on another lattice point. Is it possible for Sydney to reach $(2021,2022)$?

Lattice points are points in the Cartesian plane where both coordinates are integers.

Solution

We claim that it is impossible for Sydney to reach $(2021,2022)$.

Call a lattice point *even* if the sum of its coordinates is even, and *odd* if the sum of its coordinates is odd. For a lattice point to be even, either both coordinates must be even or both coordinates must be odd. For a lattice point to be odd, one coordinate must be even and the other coordinate must be odd.

We claim that if Sydney starts at an even lattice point and is only allowed to step on lattice points, then she can only reach even lattice points. To see this, suppose otherwise. Then there is an even lattice point, without loss of generality $(0,0)$, that can reach an odd lattice point (a,b). The midpoint of the line segment connecting these points is $(a/2,b/2)$, and hence the equation of the segment's perpendicular bisector must be

$$y - b/2 = (-a/b)(x - a/2).$$

This rearranges to

$$2(ax + by) = a^2 + b^2.$$

But since (a,b) is odd, exactly one of a,b is odd and the other is even, hence the right-hand side is odd. This means that the left-hand side is odd, meaning that $ax + by$ cannot be an integer. Thus, any point (x,y) lying on the line cannot be a lattice point. Since Sydney can move only by reflecting her position over a line that contains lattice points, Sydney cannot move from an even lattice point to an odd lattice point.

Since $(0,0)$ is even and $(2021,2022)$ is odd, it is impossible for Sydney to reach $(2021,2022)$.

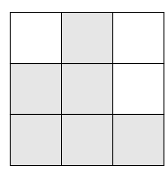
□

3/3/33. Let n be a positive integer. Let S be the set of n^2 cells in an $n \times n$ grid. Call a subset T of S a **double staircase** if

1. T can be partitioned into n horizontal nonoverlapping rectangles of dimensions 1×1, 1×2, ..., $1 \times n$, and
2. T can also be partitioned into n vertical nonoverlapping rectangles of dimensions 1×1, 2×1, ..., $n \times 1$.

In terms of n, how many double staircases are there? (Rotations and reflections are considered distinct.)

An example of a double staircase when $n = 3$ is shown below.



Solution

The answer is 4^{n-1}. Call a permutation of $1, 2, \ldots, k$ a *mountain* if no three consecutive elements a, b, c satisfy $b = \min(a, b, c)$. Let x_k be the number of mountains with k numbers. Our first claim is that $x_n = 2^{n-1}$, and our second claim is that the answer to this problem is x_n^2.

We start with the first claim that $x_n = 2^{n-1}$. It is clear that $x_1 = 1$; we will now prove that $x_k = 2x_{k-1}$ for all $k \geq 2$. In a mountain with k elements, the 1 must be first or last, which has 2 choices. The remaining $k - 1$ elements form a mountain when all the numbers are decreased by one. So $x_k = 2x_{k-1}$, which together with $x_1 = 1$ means $x_n = 2^{n-1}$.

Now we show the second claim that there are x_n^2 double staircases. The existence of a horizontal $1 \times n$ rectangle means all n columns are nonempty, and similarly the vertical $n \times 1$ rectangle means all n rows are nonempty. So the n horizontal rectangles are all in different rows, and similarly the n vertical rectangles are all in different columns.

Let H be the sequence of horizontal rectangle sizes by row, and similarly define V. We claim both H and V are mountains. Suppose H contains three elements a, b, c with $b = \min(a, b, c)$. Without loss of generality, we assume the n in H comes before b. Because $c > b$, there exists a column containing a cell of the $1 \times c$ rectangle but not the $1 \times b$ rectangle. This column contains a cell of the $1 \times n$ rectangle too, which means the column contains a cell, a gap, and a cell in order (not necessarily consecutive). This contradicts our previous conclusion that each column contains a single vertical rectangle tile. Therefore H is a mountain, and similarly V is a mountain.

For our final step, we prove any pair H and V of mountains gives exactly one double staircase. The fact that each row contains one horizontal tile and each column contains one vertical tile makes it clear at most one such subset exists. To show a double staircase exists for any mountains H and V, let T contain the squares whose row and column sizes form H and V sum to at least $n+1$. For any k with $1 \le k \le n$, there are exactly k numbers m with $1 \le m \le n$ and $m + k \ge n + 1$, so all the row and column sizes will be correct. Because H and V are mountains, for any k the numbers $n, n-1, n-2, \dots, n-k+1$ form a consecutive block in H and V in some order, so each row or column can be formed from a single horizontal or vertical tile. This shows that T works. Therefore, exactly one double staircase exists given any pair of mountains H and V, so the number of double staircases is $x_n^2 = 4^{n-1}$.

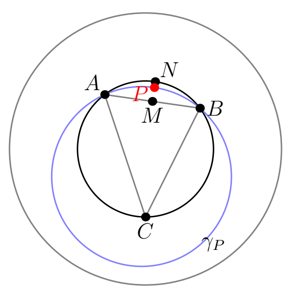
$\square$

4/3/33. Let ABC be a triangle whose vertices are inside a circle Ω. Prove that we can choose two of the vertices of ABC such that there are infinitely many circles ω that satisfy the following properties:

1. ω is inside of Ω,
2. ω passes through the two chosen vertices, and
3. the third vertex is in the interior of ω.

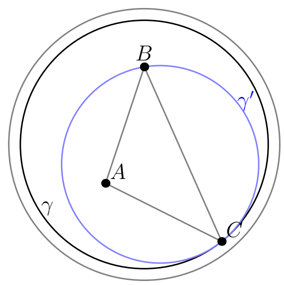
Solution

Let O be the center of Ω. Suppose the circumcircle of ABC is completely contained within Ω. Without loss of generality let minor arc AB of the circumcircle of ABC not contain point C. Let M be the midpoint of AB, let N be the midpoint of minor arc AB, let P be a point on the open segment MN, and let γ_P denote the circumcircle of ABP. Since C is on the circumcircle of ABN, we know that C is in the interior of the circumcircle of γ_P (i.e. $\angle APB + \angle ACB > 180°$). When $P = N$ we know γ_P is completely contained in Ω and taking P to be any point sufficiently close to N will have the same result by continuity.



Now we consider the case when the circumcircle is not completely contained in Ω. Without loss of generality, we assume that $AO, BO \leq CO$. Consider the circle γ centered at O passing through C. If either A or B is on γ, then the other is strictly inside γ and we proceed exactly as in the next case.

Let γ_r be the image of γ under the homothety centered at C with factor r. As r shrinks from 1 toward 0, eventually γ_r will first intersect exactly one of A or B (it cannot intersect both simultaneously or the circumcircle of ABC would be completely inside of Ω). Call this circle γ'. Without loss of generality we will assume γ' passes through C and B.



Then we can see that γ' is one circle satisfying the conditions of the problem. Let P be the midpoint of minor arc BC on γ'. Then, by adjusting P slightly on the perpendicular bisector of BC, we can ensure the resulting circumcircles of BCP do not intersect Ω and contain A.

$\square$

5/3/33. Let a, b, c, d be positive real numbers. Prove that d is an integer **if and only if** there are positive real numbers e, f satisfying

$$\left\lfloor \frac{\left\lfloor \frac{x+a}{b} \right\rfloor + c}{d} \right\rfloor = \left\lfloor \frac{x+e}{f} \right\rfloor$$

for all real numbers x. (For a real y, $\lfloor y \rfloor$ is the greatest integer less than or equal to y.)

Solution

First, we will show that if e, f exist, then d is a positive integer. We start this by showing that $f = bd$. Let $x = Nb - a$ for a variable integer N. Then the condition becomes

$$\left\lfloor \frac{N}{d} + \frac{c}{d} \right\rfloor = \left\lfloor \frac{N}{f/b} + \frac{e-a}{f} \right\rfloor.$$

For $\lfloor y \rfloor = \lfloor z \rfloor$ to be true, we must have $|y - z| < 1$. Therefore, for all integers N, we have

$$\left| \frac{N}{d} + \frac{c}{d} - \frac{N}{f/b} - \frac{e-a}{f} \right| < 1.$$

Simplifying,

$$\left| N\left(\frac{1}{d} - \frac{b}{f}\right) + \frac{c}{d} - \frac{e-a}{f} \right| < 1.$$

If $\frac{1}{d} \neq \frac{b}{f}$, then we can choose an N for which the expression inside the absolute value is as large as we want, which would violate the inequality. Therefore, $\frac{1}{d} = \frac{b}{f}$, or $f = bd$, as desired.

For k an integer, let $L(k)$ be the set of x for which the left side of the original expression is equal to k, and similarly define $R(k)$ for the right side. Observe that no matter what e we choose, for any k, $R(k)$ is always an interval of length f, or rather length bd. Therefore, $L(k)$ must also be an interval of length bd for every k.

On the other hand, $L(k)$ is the set of values for which

$$kd \leq \left\lfloor \frac{x+a}{b} \right\rfloor + c < kd + d.$$

Or, equivalently

$$0 \leq \left\lfloor \frac{x+a}{b} \right\rfloor + c - kd < d.$$

The floor term in the middle equals a particular integer m exactly when x is in an interval of length b, regardless of m and a. So the length of the interval $L(k)$ is b times the number of integers m for which the above inequality holds. But this length must also equal bd.

Therefore, d must be an integer, as desired.

Now we do the other direction: we show e, f exist if d is a positive integer. We start with the following lemma.

Lemma: If q is a positive integer and r is a real number, then for all integers n, we have

$$\left\lfloor \frac{n+r}{q} \right\rfloor = \left\lfloor \frac{n+\lfloor r \rfloor}{q} \right\rfloor.$$

Proof: The left side is equal to k exactly when

$$kq \leq n + r < kq + q.$$

Because kq and $kq+q$ are both integers, only the integer part of the middle expression matters for determining whether the inequality is satisfied. So the inequality above is equivalent to

$$kq \leq \lfloor n + r \rfloor < kq + q.$$

We simplify the middle expression:

$$kq \leq n + \lfloor r \rfloor < kq + q.$$

This is exactly the condition for the right side to equal k. So both sides of the equation are the same, and the lemma is proven.

Instead of choosing e, f now, we will derive the equation we want from the ground up and find the values at the end. To start, for any real x, there exists an integer k and a real number r with $0 \leq r < b$ such that $x = kb - a + r$. This implies that

$$k = \left\lfloor \frac{x+a}{b} \right\rfloor.$$

Next, observe that since $0 \leq r/b < 1$, we have $\left\lfloor \lfloor c \rfloor + \frac{r}{b} \right\rfloor = \lfloor c \rfloor$ for all such r. So we have that for all integers k and r with $0 \leq r < b$ that

$$\left\lfloor \frac{k + \lfloor c \rfloor}{d} \right\rfloor = \left\lfloor \frac{k + \left\lfloor \lfloor c \rfloor + \frac{r}{b} \right\rfloor}{d} \right\rfloor.$$

Applying the lemma to both sides of the equation, we have

$$\left\lfloor \frac{k + c}{d} \right\rfloor = \left\lfloor \frac{k + \lfloor c \rfloor + \frac{r}{b}}{d} \right\rfloor.$$

This implies that

$$\left\lfloor \frac{k + c}{d} \right\rfloor = \left\lfloor \frac{kb - a + r + b\lfloor c \rfloor + a}{bd} \right\rfloor.$$

Recall that both $x = kb - a + r$ and $k = \left\lfloor \frac{x+a}{b} \right\rfloor$. We substitute both identities into the above, obtaining that for all x,

$$\left\lfloor \frac{\left\lfloor \frac{x+a}{b} \right\rfloor + c}{d} \right\rfloor = \left\lfloor \frac{x + b\lfloor c \rfloor + a}{bd} \right\rfloor .$$

This is the equation we wanted if we set $f = bd$ and $e = b\lfloor c \rfloor + a$. The proof is now complete. $\square$

Problems by Thomas Lam, Kevin Ren, and USAMTS Staff.