USA Mathematical Talent Search
Round 1 Solutions
Year 33 — Academic Year 2021–2022
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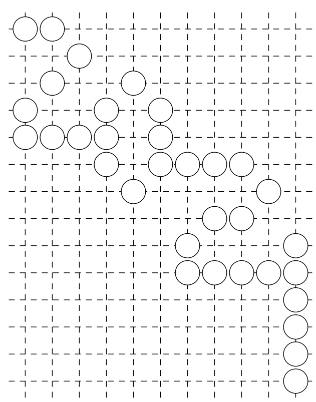
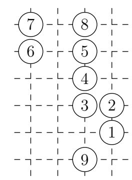
1/1/33. 33 counters are shown in the left grid below. Choose a counter to start at and remove it from the grid. At each subsequent step, choose a direction (up, down, left, or right), move along the grid line from your current position to the nearest counter in that direction, and remove that counter. You cannot choose a direction that reverses your previous one (e.g., left then right is not allowed). Your goal is to pick up all 33 counters in a single sequence of steps. When you find the right sequence, write the numbers 1 to 33 on the counters so that N is written on the N^{th} counter you removed.

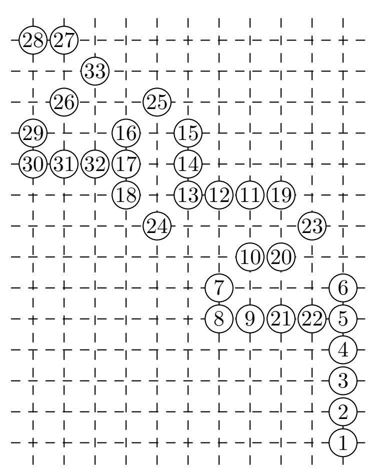
A smaller example of a solved grid is shown to the right below. (Note that the final move from 8 to 9 is possible because counters 3, 4, and 5 have been removed in earlier steps.)

There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)





Solution



2/1/33. Find, with proof, the minimum positive integer n with the following property: for any coloring of the integers $\{1, 2, \ldots, n\}$ using the colors red and blue (that is, assigning the color "red" or "blue" to each integer in the set), there exist distinct integers a, b, c between 1 and n, inclusive, all of the same color, such that $2a + b = c$.

Solution

Call a triple (a, b, c) *good* if $2a + b = c$ and all of a, b, c are the same color.

We can color $\{1, \ldots, 14\}$ as 1–3 red, 4–12 blue, and 13–14 red, and there is no good triple:

- Choosing $\{a, b\} \in \{1, 2, 3\}$ gives $4 \leq 2a + b \leq 8$, so a and b are red but $c = 2a + b$ is blue.
- Choosing $\{a, b\} \in \{4, 5, \ldots, 12\}$ gives $2a + b \geq 13$, so a and b are blue but $c = 2a + b$ is either red or too big.
- Choosing either $a \geq 13$ or $b \geq 13$ gives $2a + b \geq 15$, so $c = 2a + b$ is too big.

Thus we must have $n > 14$ to have a good triple.

We prove that any coloring of $\{1, \ldots, 15\}$ produces a good triple, by contradiction. Suppose we have a coloring of $\{1, \ldots, 15\}$ for which there is no good triple. We have two cases:

Case 1: 1 and 2 are the same color. Without loss of generality, suppose 1 and 2 are both red. Using our assumption that there are no good triples, the following colors are required:

- 4 must be blue, because otherwise $(1, 2, 4)$ would be a red good triple.
- 5 must be blue, because otherwise $(2, 1, 5)$ would be a red good triple.
- 13 must be red, because otherwise $(4, 5, 13)$ would be a blue good triple.
- 15 must be blue, because otherwise $(1, 13, 15)$ would be a red good triple.
- 7 must be red, because otherwise $(4, 7, 15)$ would be a blue good triple.
- 3 must be blue, because otherwise $(2, 3, 7)$ would be a red good triple.
- 9 must be blue, because otherwise $(1, 7, 9)$ would be a red good triple.

And now we have a contradiction, because $(3, 9, 15)$ is a blue good triple.

Case 2: 1 and 2 are different colors. Without loss of generality, suppose 1 is red and 2 is blue. If any $5 \le x \le 13$ is colored red, then since the triple $(1, x, x+2)$ cannot be a red good triple, we see that $x + 2$ must be colored blue. But then, since the triple $(2, x-2, x+2)$ cannot be a blue good triple, we see that $x - 2$ must be colored red. But then $(1, x-2, x)$ is a red good triple, which is not allowed. So all of $\{5, \dots, 13\}$ must be colored blue. And now we have a contradiction: in particular $(2, 5, 9)$ is a good blue triple.

So, by contradiction, any coloring of $\{1, \dots, 15\}$ must produce a good triple, and therefore our final answer is $n = \boxed{15}$. □

3/1/33. Let S be a subset of $\{1, 2, \ldots, 500\}$ such that no two distinct elements of S have a product that is a perfect square. Find, with proof, the maximum possible number of elements in S.

Solution

Let $f(n)$ denote the square-free part of n: that is, $f(n) = \frac{n}{d}$, where d is the greatest divisor of n that is a perfect square. We note that mn is a perfect square if and only if $f(m) = f(n)$. In particular, every element $s \in S$ must have a different value of $f(s)$. So S is in 1-1 correspondence (via f) with a subset of T, where T is the set of square-free positive integers up to 500. Thus $|S| \le |T|$, and choosing $|S| = |T|$ provides a maximal example.

So it remains to compute $|T|$. It's easier to count non-square-free integers up to 500: these are multiples of perfect squares greater than 1. We count multiples of p^2 for p prime:

There are $\lfloor \frac{500}{4} \rfloor = 125$ multiples of $2^2 = 4$.

There are $\lfloor \frac{500}{9} \rfloor = 55$ multiples of $3^2 = 9$. However, $\lfloor \frac{55}{4} \rfloor = 13$ of these are also multiples of 4, which we have already counted. Thus, we get $55 - 13 = 42$ new numbers that are multiples of 9.

There are $\lfloor \frac{500}{25} \rfloor = 20$ multiples of $5^2 = 25$. However, $\lfloor \frac{20}{4} \rfloor = 5$ of these are also multiples of 4, and $\lfloor \frac{20}{9} \rfloor = 2$ of these are also multiples of 9. Thus, we get $20 - 5 - 2 = 13$ new numbers that are multiples of 25.

There are $\lfloor \frac{500}{49} \rfloor = 10$ multiples of $7^2 = 49$. However, $\lfloor \frac{10}{4} \rfloor = 2$ of these are also multiples of 4, and $\lfloor \frac{10}{9} \rfloor = 1$ of these is also a multiple of 9. Thus, we get $10 - 2 - 1 = 7$ new numbers that are multiples of 49.

There are $\lfloor \frac{500}{121} \rfloor = 4$ multiples of $11^2 = 121$. However, $\lfloor \frac{4}{4} \rfloor = 1$ of these is also a multiple of 4. Thus, we get $4 - 1 = 3$ new numbers that are multiples of 121.

There are $\lfloor \frac{500}{169} \rfloor = 2$ multiples of $13^2 = 169$.

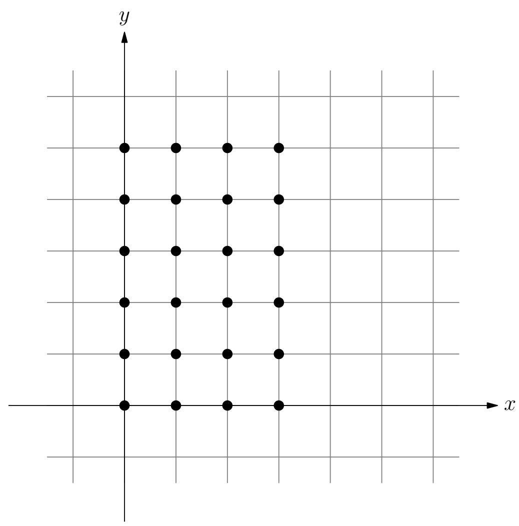
There is $\lfloor \frac{500}{289} \rfloor = 1$ multiple of $17^2 = 289$.

There is $\lfloor \frac{500}{361} \rfloor = 1$ multiple of $19^2 = 361$.

The next prime square is $23^3 = 529 > 500$, which is too large, so these are all the multiples of prime squares.

So our final answer is $|T| = 500 - (125 + 42 + 13 + 7 + 3 + 2 + 1 + 1) = 500 - 194 = \boxed{306}$. $\square$

4/1/33. Let m, n, k be positive integers such that $k \leq mn$. Let S be the set consisting of the $(m+1)$-by-$(n+1)$ rectangular array of points on the Cartesian plane with coordinates (i, j) where i, j are integers satisfying $0 \leq i \leq m$ and $0 \leq j \leq n$. The diagram below shows the example where $m = 3$ and $n = 5$, with the points of S indicated by black dots:



Prove that there exist points A, B, C in S such that the area of $\triangle ABC$ is $\frac{k}{2}$.

Solution

Let $A = (0, 1)$ and $B = (m, 0)$. If we let $D = (k, 1)$, then $[ABD] = \frac{k}{2}$ (using $AD = k$ as the base and the height from B to $\overline{AD}$ of 1), but if $k > m$ then $D \notin S$.

Note that the slope of $\overline{AB}$ is $-\frac{1}{m}$, so if $C = (k - mt, 1 + t)$ for any t, then $\overline{CD} \parallel \overline{AB}$, and hence $[ABC] = [ABD] = \frac{k}{2}$, as desired.

So if suffices to prove that we can always choose such a t so that $C \in S$, which means that our t must satisfy

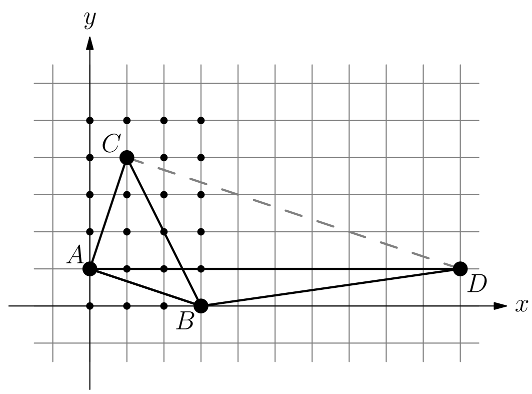
$$0 \leq k - mt \leq m, \tag{1}$$
$$0 \leq 1 + t \leq n. \tag{2}$$

We claim that the choice $t = \lfloor \frac{k-1}{m} \rfloor$ works. (Note that if $k \leq m$, then $t = 0$ and $C = D$, which works since because in this case $D \in S$ to begin with.)

Since $1 \leq k \leq mn$, we have $0 \leq t \leq n - 1$, and hence $1 \leq 1 + t \leq n$, which gives (2).

Then, by the definition of the floor function we have $\frac{k-1}{m} - 1 < t \leq \frac{k-1}{m}$. Thus we have $k - 1 - m < mt \leq k - 1$, so $1 \leq k - mt < m + 1$, which gives (1).

The picture below shows an example of the construction for $m = 3$, $n = 5$, and $k = 10$. Note that $D = (10, 1)$ gives triangle ABD with height 1 and base $AD = 10$, so $[ABD] = 5$. Our construction above gives $t = \lfloor \frac{10-1}{3} \rfloor = 3$, so $C = (10 - 3 \cdot 3, 1 + 3) = (1, 4)$ satisfies $\overline{CD} \parallel \overline{AB}$, so that $[ABC] = [ABD] = 5$. One can further check that applying Pick's Theorem also verifies that $[ABC] = 4 + \frac{4}{2} - 1 = 5$.



5/1/33. Define a sequence of positive rational numbers $x_0, x_1, x_2, x_3, \ldots$ by $x_0 = 2$, $x_1 = 3$, and for all $n \geq 2$,

$$x_n = \frac{x_{n-1}^2 + 5}{x_{n-2}}.$$

(a) Prove that x_n is an integer for all $n \geq 0$.
(b) Prove that if x_n is prime, then either $n = 0$ or $n = 2^k$ for some integer $k \geq 0$.

Solution

(a) Rewrite the recurrence as $x_n x_{n-2} - x_{n-1}^2 = 5$. This is true for all $n \geq 2$, so

$$x_n x_{n-2} - x_{n-1}^2 = x_{n+1}x_{n-1} - x_n^2.$$

We can rearrange this as

$$x_n x_{n-2} + x_n^2 = x_{n+1}x_{n-1} + x_{n-1}^2,$$

and dividing by $x_n x_{n-1}$ yields

$$\frac{x_{n-2} + x_n}{x_{n-1}} = \frac{x_{n+1} + x_{n-1}}{x_n}.$$

Thus this quantity is constant for all $n \geq 2$. In particular, since $x_2 = \dfrac{x_1^2 + 5}{x_0} = \dfrac{3^2 + 5}{2} = 7$ and $\dfrac{x_2 + x_0}{x_1} = \dfrac{7 + 2}{3} = 3$, we have

$$\frac{x_{n-2} + x_n}{x_{n-1}} = 3$$

for all $n \geq 2$. Therefore $x_n = 3x_{n-1} - x_{n-2}$ for all $n \geq 2$, and we see that all terms in the sequence must be integers. We also note that the sequence is strictly increasing and in particular all elements of the sequence are greater than 1. $\square$

(b) We can rewrite the recurrence as $x_{n-2} = 3x_{n-1} - x_n$. Note that this is the same recurrence but in the opposite direction, and this allows us to extend the sequence to x_n for $n < 0$. In particular, note that $x_{-1} = 3x_0 - x_1 = 3(2) - 3 = 3$, so since $x_{-1} = x_1 = 3$, we have $x_{-n} = x_n$ for all n.

Lemma: Let p be prime and n, r be integers, with $r > 0$, such that x_n and x_{n+r} are both multiples of p. Then x_{n+mr} is a multiple of p for any integer m.

Proof: Let $x_{n+1} \equiv a \pmod{p}$ and $x_{n+r+1} \equiv b \pmod{p}$. If $a \equiv 0$ then there is nothing to prove: every subsequent element is a multiple of p. Otherwise, by induction, for all $k \geq 0$, $x_{n+k} \equiv c_k a \pmod{p}$ and $x_{n+r+k} \equiv c_k b \pmod{p}$ for some c_k. (This is because the recurrence relation is linear mod p.) But when $k = r$, we must have $c_r \equiv 0 \pmod{p}$, since $x_{n+r} \equiv 0 \pmod{p}$, and thus $x_{n+2r} \equiv 0 \pmod{p}$ as well. The full result then follows by straightforward induction. $\square$

Applying this to x_{-n} and $r = 2n$, we get that if x_n is a multiple of p, then so is x_{mn} for any odd integer m. Considering this fact for all primes, we see that $x_n | x_{mn}$ for any odd integer m.

In particular, if $n = m2^k$ for some odd m, then $x_{2^k} | x_n$. So if $m > 1$, then x_n is a nontrivial multiple of $x_{2^k} > 1$, and thus must be composite.

Hence, x_n is prime only if $n = 0$ or $n = 2^k$ for some k. $\square$

Note 1: The condition is certainly not sufficient for x_{2^k} to be prime. One can check that

$$(x_0, x_1, x_2, x_4, x_8) = (2, 3, 7, 47, 2207)$$

are all prime, but $x_{16} = 4870847 = 1087 \cdot 4481$ is composite. Indeed, it is an open question if x_{2^k} is prime for *any* $k > 3$: there are no known examples of x_{2^k} prime for $k > 3$, but it is unproven that they must all be composite.

Note 2: It can be shown that $x_n = L_{2n}$, where L_k is the k^{th} *Lucas number*, defined by the recurrence $L_0 = 2$, $L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for all $n \geq 2$. (Note that this is the same recurrence relation as the Fibonacci numbers, but with different initial conditions.) A solution is acceptable if it proves this fact and then uses known facts about Lucas numbers, cited from a reputable source, to complete the proof.