USA Mathematical Talent Search

Round 3 Solutions

Year 32 — Academic Year 2020–2021

www.usamts.org

1/3/32. Place the 21 two-digit prime numbers in the white squares of the grid on the right so that each two-digit prime is used exactly once. Two white squares sharing a side must contain two numbers with either the same tens digit or ones digit. A given digit in a white square must equal at least one of the two digits of that square's prime number.

There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the constraints above. (Note: in any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

3	1	3	2	
	■			
9	■	9	■	5
			■	
	4	1	1	3

Solution

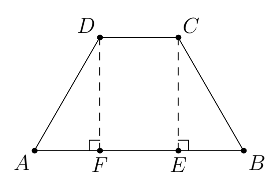
^{3} 31	^{1} 71	^{3} 73	^{2} 23	83
37	■	79	29	89
^{9} 97	■	^{9} 19	■	^{5} 59
67	47	17	■	53
61	^{4} 41	^{1} 11	^{1} 13	^{3} 43

2/3/32. Find distinct points A, B, C, and D in the plane such that the length of the segment AB is an even integer, and the lengths of the segments AC, AD, BC, BD, and CD are all odd integers. In addition to stating the coordinates of the points and distances between points, please include a brief explanation of how you found the configuration of points and computed the distances.

Solution

Many answers are possible. We provide three examples of constructions that work, all of which revolve around a triangle with side lengths $3, 5$, and 7, which has an interior angle of $120°$. As for how to find this triangle, the Law of Cosines tells us that any triangle with integer lengths must have interior angles with rational cosines. We might first consider using right triangles for our construction, but since every primitive Pythagorean triple has one even-length leg, we would end up with too many even lengths. This leads us to look for triangles with $60°$ or $120°$ angles.

Construction 1. Consider the isosceles trapezoid $ABCD$ with vertices at $A = (0, 0), B = (8, 0), C = (\frac{11}{2}, \frac{5\sqrt{3}}{2}), D = (\frac{5}{2}, \frac{5\sqrt{3}}{2})$. Let E and F be the feet of the perpendiculars from points C and D respectively to AB.

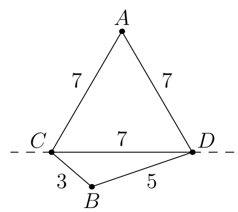


We have $AB = 8$ and $CD = 3$. Also, $CE = DF = \frac{5\sqrt{3}}{2}$ and $AF = BE = \frac{5}{2}$, so $\triangle ADF$ and $\triangle BCE$ are $30-60-90$ right triangles, and $AD = BC = 5$. Since $ABCD$ is an isosceles trapezoid (or by using the angles in the previously mentioned triangle), we know that its opposite angles are supplementary, so it must be a cyclic quadrilateral. Therefore, Ptolemy's Theorem tells us that

$$\begin{aligned} AC \cdot BD &= AB \cdot CD + AD \cdot BC \\ &= 8 \cdot 3 + 5 \cdot 5 \\ &= 49. \end{aligned}$$

Since $ABCD$ is isosceles, by symmetry we have $AC = BD$, which means that $AC = BD = 7$. Hence, the vertices of $ABCD$ have the desired property.

Construction 2. Let $\triangle ACD$ be an equilateral triangle with side length 7. Consider that the line CD divides the plane into two half-planes and that the Triangle Inequality tells us that a triangle with side lengths $3, 5$, and 7 is nondegenerate. Hence, it is possible to place a point B so that $BC = 3, BD = 5$, and B is in the half-plane determined by BD that does not contain A.



We claim that $ACBD$ is cyclic: Using the Law of Cosines on $\triangle BCD$, we see that

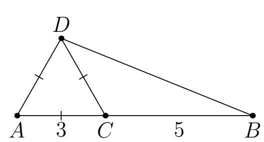
$$\begin{aligned} CD^2 &= BC^2 + BD^2 - 2 \cdot BC \cdot BD \cdot \cos(\angle CBD) \\ \Leftrightarrow 7^2 &= 3^2 + 5^2 - 2 \cdot 3 \cdot 5 \cdot \cos(\angle CBD) \\ \Leftrightarrow -\frac{1}{2} &= \cos(\angle CBD), \end{aligned}$$

which tells us that $\angle CBD = 120°$. Since $\triangle ACD$ is equilateral, we have $\angle CAD = 60°$, so the opposite angles of $ACBD$ are supplementary and hence $ACBD$ is cyclic. Therefore, by Ptolemy's Theorem, we have

$$\begin{aligned} AB \cdot CD &= AC \cdot BD + AD \cdot BC \\ \Leftrightarrow AB \cdot 7 &= 7 \cdot 5 + 7 \cdot 3 \\ \Leftrightarrow AB &= 8. \end{aligned}$$

Hence, the vertices of $ACBD$ have the desired property.

Construction 3. Let AB be a line segment with length 8. Let C lie on AB so that $AC = 3$ and $BC = 5$, and let D be a point so that $\triangle ACD$ is equilateral. (Two such points must always exist, but since we need only provide a construction here, consider that the points $A = (0, 0), B = (8, 0), C = (3, 0)$, and $D = (\frac{3}{2}, \frac{3\sqrt{3}}{2})$ satisfy this.)



From this, we immediately see that $AB = 8, BC = 5$, and $AC = AD = CD = 3$. To compute BD, note that since $\triangle ACD$ is equilateral and B lies on the extension of AC, we have $\angle BCD = 120°$, so by the Law of Cosines we have

$$\begin{aligned} BD^2 &= BC^2 + CD^2 - 2 \cdot BC \cdot CD \cdot \cos(\angle BCD) \\ &= 5^2 + 3^2 - 2 \cdot 5 \cdot 3 \cdot \cos(120°) \\ &= 49, \end{aligned}$$

so $BD = 7$. Hence, the vertices A, B, C, D have the desired property.

3/3/32. Find, with proof, all positive integers n with the following property: There are only finitely many positive multiples of n which have exactly n positive divisors.

Solution

We claim that the only positive integers n with the desired property are $n = 1$, $n = 4$, and all other squarefree integers n (that is, all other integers not divisible by the square of a prime number).

First, we first show that each of these n have the desired property. For $n = 1$, there is only one multiple of 1 with exactly 1 positive divisor, namely 1 itself. For $n = 4$, any positive integer with exactly 4 divisors must take the form pq or p^3, where p and q are distinct primes; since 4 is a power of 2, the only multiple of 4 which takes either of these forms is $2^3 = 8$, so $n = 4$ indeed has the desired property.

Now suppose $n \geq 2$ is a squarefree integer. Then n must be the product of distinct primes, so let $n = p_1 p_2 \cdots p_k$, where all p_i are distinct, and suppose m is a multiple of n with n divisors. Let

$$m = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k} \cdot m'$$

be the prime factorization of m, where $\gcd(m', n) = 1$. By counting the number of divisors of m, we have

$$(e_1 + 1)(e_2 + 1) \cdots (e_k + 1) \cdot n' = n = p_1 p_2 \cdots p_k,$$

where n' is the number of divisors of m'. The right-hand side of this equation has exactly k prime divisors, counting with multiplicity; however, each term $e_i + 1 \geq 2$ has at least one prime divisor. It follows that $n' = 1$, so m' only has one divisor and therefore $m' = 1$. Hence, the numbers $e_1 + 1, e_2 + 1, \ldots, e_k + 1$ must be a permutation of the numbers $p_1, p_2, \ldots, p_k$. So there are $k!$ possible values for m, and thus only finitely many positive multiples of n which have exactly n positive divisors. This shows that $n = 1, n = 4$, and all other squarefree integers n have the desired property, and it remains to show these are the only such n.

Finally, suppose that $n \neq 4$ and n is not squarefree. We will show there are infinitely many positive multiples of n with exactly n positive divisors. Since n is not squarefree, it is divisible by the square of some prime. Suppose first that $p^2 \mid n$, where p is some odd prime. (We will deal with the case of $p = 2$ later.) Let

$$n = p^e \cdot p_1^{e_1} \cdot p_2^{e_2} \cdots p_k^{e_k}$$

be the prime factorization of n, where $e \geq 2$. Given any prime q distinct from $p, p_1, p_2, \ldots, p_k$, let

$$m_q = p^{p^{e-1}-1} \cdot p_1^{p_1^{e_1}-1} p_2^{p_2^{e_2}-1} \cdots p_k^{p_k^{e_k}-1} \cdot q^{p-1}.$$

We claim that m_q is a positive multiple of n with exactly n positive divisors: First, note that m_q has exactly

$$\begin{aligned}(p^{e-1}-1+1)\cdot(p_1^{e_1}-1+1)\cdot(p_2^{e_2}-1+1)\cdots(p_k^{e_k}-1+1)\cdot(p-1+1)\\ &= (p^{e-1})\cdot(p_1^{e_1})\cdot(p_2^{e_2})\cdots(p_k^{e_k})\cdot p\\ &= p^e\cdot p_1^{e_1}\cdot p_2^{e_2}\cdots p_k^{e_k}\\ &= n\end{aligned}$$

divisors. To show that m_q is a multiple of n, we claim that the exponents of $p, p_1, \ldots, p_k$ in the prime factorization of n are less than or equal to that of n, and show this through proving a more general lemma:

Lemma. If $a \geq 2$ and $b \geq 1$ are integers and $(a,b) \neq (2,1)$, then $a^b \geq b+2$.

Proof. If $b=1$, then we must have $a \geq 3$, and so $a^b = a \geq 3 = b+2$. If $b \geq 2$ (and $a \geq 2$) then we have

$$\begin{aligned}a^b &\geq 2^b\\ &= \binom{b}{0}+\binom{b}{1}+\binom{b}{2}+\cdots+\binom{b}{b}\\ &\geq \binom{b}{0}+\binom{b}{1}+\binom{b}{b}\\ &= b+2,\end{aligned}$$

as desired. □

Now, recall that

$$m_q = p^{p^{e-1}-1}\cdot p_1^{p_1^{e_1}-1}p_2^{p_2^{e_2}-1}\cdots p_k^{p_k^{e_k}-1}\cdot q^{p-1}.$$

Since $p \geq 3$ and $e \geq 2$, our Lemma tells us that $p^{e-1} \geq e+1$. Likewise, since each $p_i \geq 2$ and $e_i \geq 1$, we have $p_i^{e_i} \geq e_i + 1$. Therefore, it follows that m_q is a multiple of n. However, since there are infinitely many primes, there are infinitely many choices for q, so there are infinitely many possible multiples m_q of n with exactly n positive divisors.

It remains to consider the case in which $n \neq 4$ and 2 is the only prime whose square divides n. In this case, we must either have $n = 2^e$ for some $e \geq 3$, or $n = 2^e \cdot p_1 \cdot p_2 \cdots p_k$, where $e \geq 2$ and the p_i's are distinct odd primes. In the first case, given any odd prime q, let

$$m_q = 2^{2^{e-1}-1}\cdot q.$$

This has $(2^{e-1}-1+1)\cdot 2 = 2^e = n$ divisors. Furthermore, as our Lemma tells us that $2^{e-1} \geq e+1$, this is a multiple of n.

In the second case, we consider two subcases: $e = 2$ and $e > 2$. If $e = 2$, for any odd prime q distinct from all the p_i's, let

$$m_q = 2^{p_1-1} \cdot p_1 \cdot p_2^{p_2-1} \cdots p_k^{p_k-1} \cdot q.$$

Since $p_i \geq 3$ for all $1 \leq i \leq k$, this is a multiple of $n = 2^2 \cdot p_1 \cdot p_2 \cdots p_k,$. Additionally, the number of divisors is $p_1 \cdot 2 \cdot p_2 \cdots p_k \cdot 2 = n$.

If $e > 2$, then for any odd prime q distinct from all the p_i's, let

$$m_q = 2^{2^{e-1}-1} \cdot p_1^{p_1-1} \cdot p_2^{p_2-1} \cdots p_k^{p_k-1} \cdot q.$$

This is a multiple of n, and it has

$$\begin{aligned}(2^{e-1} - 1 + 1) \cdot (p_1 - 1 + 1) \cdot (p_2 - 1 + 1) \cdots (p_k - 1 + 1) \cdot (1 + 1)\\ &= 2^{e-1} \cdot p_1 \cdot p_2 \cdots p_k \cdot 2\\ &= 2^e \cdot p_1 \cdot p_2 \cdots p_k\\ &= n\end{aligned}$$

divisors.

In all cases, since there are infinitely many primes, there are infinitely many choices for q, and hence infinitely many positive multiples m_q of n with exactly n positive divisors, as desired.

4/3/32. In a group of $n > 20$ people, there are some (at least one, and possibly all) pairs of people that know each other. Knowing is symmetric; if Alice knows Blaine, then Blaine also knows Alice. For some values of n and k, this group has a peculiar property: If any 20 people are removed from the group, the number of pairs of people that know each other is at most $\dfrac{n-k}{n}$ times that of the original group of people.
(a) If $k = 41$, for what positive integers n could such a group exist?
(b) If $k = 39$, for what positive integers n could such a group exist?

Solution

We claim that if $k = 41$, such a group could not exist for any positive integer n, and if $k = 39$, such a group could exist for any positive integer $n \geq 381$. To show this, we present a solution making use of graph theory. For more background, we invite the reader to consult an external introductory reference on the subject, e.g. Miklós Bóna's *A Walk Through Combinatorics* or Oscar Levin's *Discrete Mathematics*.

Suppose such a group does exist; we will investigate the conditions under which this could happen. Let G be the graph generated by this group of people and the acquaintances within the group, where each vertex represents a person and two vertices are connected by an edge if and only if the two respective people know each other. Let $V(G)$ and $E(G)$ be the set of vertices and edges of G respectively, so that $|V(G)| = n$ and $|E(G)|$ is the number of pairs of people that know each other.

The property this group has applies when *any* 20 people are removed from the group, but only considering what happens when we remove a few particular sets of 20 people from this group would not tell us much about the group itself. However, removing sets of 20 people in a "symmetric" fashion could shed some light on this.

To this end, consider what happens when we remove every possible set of 20 people from the group, and sum up the number of pairs of people that know each other across all these smaller groups. In terms of G, this is the number of edges of all induced subgraphs [1] given by taking a vertex subset of size $n - 20$, given by

$$T = \sum_{S \subseteq V(G), |S| = n-20} |E(G[S])|.$$

At first glance, this is not a very tractable sum, but we can also express T in terms of $|E(G)|$ by looking at how many times each edge is counted: Notice that in this sum, each edge $e \in E(G)$ is counted once for every induced subgraph $G[S]$ where $e \in E(G[S])$. By definition, $e \in E(G[S])$ if and only if both endpoints of e are in S, so we can count the

[1] By an *induced subgraph* $G[S]$, we mean the subgraph of G whose vertex set is $S \subseteq V(G)$, and whose edge set consists of all edges in $E(G)$ with both endpoints in S.

number of possible choices of $G[S]$ by counting the number of possible of choices of S. If both endpoints of e are in S, then there are $\binom{n-2}{20}$ ways to choose a subset $S \subseteq V(G)$ with $|S| = n - 20$, since we can pick such a vertex subset by excluding any 20 vertices of $V(G)$ aside from the two endpoints of e. Adding this up for every edge $e \in E(G)$, this tells us that $T = \binom{n-2}{20}|E(G)|$.

Now, the condition in our problem lets us compare values of $|E(G[S])|$ in our summation for T and $|E(G)|$: Since the number of pairs of people that know each other in any group of $n - 20$ of these people is at most $\frac{n-k}{n}$ times that of the original group of people, for any choice of $S \subseteq V(G)$ where $|S| = n - 20$ we have

$$|E(G[S])| \leq \frac{n-k}{n} \cdot |E(G)|.$$

Also, there are $\binom{n}{n-20} = \binom{n}{20}$ vertex subsets of G of size $n - 20$, so

$$\begin{aligned} T &= \sum_{S \subseteq V(G), |S|=n-20} |E(G[S])| \\ &\leq \sum_{S \subseteq V(G), |S|=n-20} \frac{n-k}{n} \cdot |E(G)| \\ &= \binom{n}{20} \cdot |E(G)| \cdot \frac{n-k}{n}. \end{aligned}$$

Substituting in our earlier expression for T, we get

$$\binom{n-2}{20} \cdot |E(G)| \leq \binom{n}{20} \cdot |E(G)| \cdot \frac{n-k}{n}.$$

This simplifies to

$$(n-20)(n-21) \leq n(n-1) \cdot \frac{n-k}{n}.$$

For part (a), when $k = 41$, this simplifies to $n \leq -379$. Since we are given that $n > 20$, we conclude that there are $\boxed{\text{no positive integers } n}$ where such a group could exist.

For part (b), when $k = 39$, this simplifes to $n \geq 381$. We claim such a group of people could exist for every such n: For a fixed $n \geq 381$, suppose there were a group of n people where everyone knew each other. Then the corresponding graph G would be the complete graph K_n (with $\binom{n}{2}$ edges), and every subgraph $G[S]$ with $S \subseteq V(G)$, $|S| = n - 20$ would be the complete graph K_{n-20} (with $\binom{n-20}{2}$ edges). But since $n \geq 381$, it follows by algebra that $\binom{n-20}{2} \leq \frac{n-39}{n} \cdot \binom{n}{2}$, which means that this group fits the conditions in the problem. Therefore, such a group exists for all $\boxed{n \geq 381}$.

5/3/32. Let $n \geq 3$ be an integer. Let f be a function from the set of all integers to itself with the following property: If the integers $a_1, a_2, \ldots, a_n$ form an arithmetic progression, then the numbers

$$f(a_1), f(a_2), \ \ldots, \ f(a_n)$$

form an arithmetic progression (possibly constant) in some order. Find all values for n such that the only functions f with this property are the functions of the form $f(x) = cx + d$, where c and d are integers.

Solution

We claim that an integer $n \geq 3$ satisfies the given condition if and only if it is composite.

First, suppose $n \geq 3$ is prime. Then the function f given by $f(x) = x \pmod{n}$ is not of the form $f(x) = cx + d$, but has the desired property: Take any n-term arithmetic sequence $(a_i) = a_1, a_2, \cdots, a_n$, and consider the sequence

$$(b_i) = f(a_1), f(a_2), \cdots, f(a_n).$$

If the common difference of (a_i) is a multiple of n, then all of its terms have the same residue mod n, so (b_i) will be constant and hence arithmetic. If the common difference of this sequence is not a multiple of n, then the terms of (b_i) will be $0, 1, 2, \cdots, n-1$ in some order, which is an arithmetic sequence.

Now, suppose that $n \geq 3$ is composite, and let f be a function with the given property. We claim that f must be linear. First, suppose that there exists an integer m such that $f(m) = f(m+n) = k$. Since n is composite, we can write $n = ab$ for some $a, b \geq 2$. Then the n-term sequence

$$f(m), f(m+a), f(m+2a), \cdots, f(m+(n-1)a)$$

is an arithmetic sequence containing both $f(m)$ and $f(m+n)$, so it must be constant. Hence, we have $f(m) = f(m+a) = k$. Now, the n-term sequence

$$f(m), f(m+1), f(m+2), \cdots, f(m+n-1)$$

contains both $f(m)$ and $f(m+a)$, so it must also be a constant arithmetic sequence. Since $n \geq 3$, the sequence $f(m+1), f(m+2), \cdots, f(m+n)$ thus contains two equal terms, so it is also constant and therefore $f(m+1) = k$. Then, by successively applying this reasoning at the sequences

$$f(m+2), f(m+3), \cdots, f(m+n+1),$$

$$f(m+3), f(m+4), \cdots, f(m+n+2),$$

$$f(m+4), f(m+5), \cdots, f(m+n+3),$$

we see that $f(m+2) = f(m+3) = f(m+4) = k$. Continuing this way, we get $f(m+s) = k$ for every $s \geq 0$. Similarly, by successively applying this reasoning to the sequences

$$f(m-1), f(m), \cdots, f(m+n-2),$$

$$f(m-2), f(m-1), \cdots, f(m+n-3),$$

$$f(m-3), f(m-2), \cdots, f(m+n-4),$$

we see that $f(m-1) = f(m-2) = f(m-3) = k$. Continuing this way, we get $f(m+s) = k$ for every $s \leq 0$. Hence, f must be constant and therefore must be linear.

Next, assume that $f(m) \neq f(m+n)$ for every integer m. We claim that for every integer m, the $n+1$ numbers

$$f(m), f(m+1), f(m+2), \cdots, f(m+n)$$

form an arithmetic sequence in some order, with $f(m)$ and $f(m+n)$ being the first and last terms of the sequence in some order. First, since f is a function with the given property, there exist some integers a and d such that $f(m), f(m+1), \cdots, f(m+n-1)$ are $a, a+d, a+2d, \cdots, a+(n-1)d$ in some order.

Now, we claim that $f(m)$ must either be the first or last term of the arithmetic sequence given by $f(m), f(m+1), \cdots, f(m+n-1)$, i.e. either $f(m) = a$ or $f(m) = a+(n-1)d$. We show this by contradiction: Suppose that $f(m) = a + kd$ for some $1 \leq k \leq n-2$. Then as $n \geq 4$ by compositeness, there must be two terms of the form $a+id, a+(i+1)d$ for some $1 \leq i \leq n-2$ among the n terms

$$f(m+1), f(m+2), \ldots, f(m+n).$$

Hence, the common difference of the sequence given by these n terms is at most d. But then some two of these terms must be equal to $a+(k-1)d$ and $a+(k+1)d$, which in turn means that $f(m+n) = a+kd$. However, this contradicts our earlier assumption that $f(m) \neq f(m+n)$ for every m, so we either have $f(m) = a$ or $f(m) = a+(n-1)d$.

Without loss of generality, let $f(m) = a$. Then since $f(m+n) \neq a$ and both

$$f(m), f(m+1), \cdots, f(m+n-1),$$

$$f(m+1), f(m+2), \cdots, f(m+n)$$

form arithmetic sequences in some order with common difference d, it follows that $f(m+n) = a+nd$, completing the proof of our claim. Note that consequently, $f(m+1), f(m+2), \cdots, f(m+n-1)$ are integers between $f(m)$ and $f(m+n)$ exclusive.

By our work above, we know there exist some integers a, d such that $f(0) = a, f(n) = a + nd$, and

$$f(0), f(1), f(2), \cdots, f(n)$$

form an arithmetic sequence with common difference d in some order; let this sequence be (a_i). Our work also tells us that

$$f(1), f(2), \cdots, f(n+1)$$

form an arithmetic sequence in some order with $f(1), f(n+1)$ being the first and last terms in some order; let this sequence be (b_i). Since (a_i) has common difference d, it follows that (b_i) must have common difference d as well. Also, since $f(m) \neq f(m+n)$ for every integer m, we either have $f(1) = a + d$ or $f(1) = a + (n+1)d$, but we know that $a + (n+1)d$ is not a term of (a_i), so $f(1) = a + d$ and $f(n+1) = a + (n+1)d$. Then, by successively applying this reasoning to the sequences

$$f(2), f(3), \cdots, f(n+2),$$

$$f(3), f(4), \cdots, f(n+3),$$

$$f(4), f(5), \cdots, f(n+4),$$

we see that $f(2) = a + 2d$, $f(3) = a + 3d$, and $f(4) = a + 4d$. Continuing this way, we see that $f(m) = a + md$ for all $m \geq 0$. Similarly, by successively applying this reasoning to the sequences

$$f(-1), f(0), \cdots, f(n-1),$$

$$f(-2), f(-1), \cdots, f(n-2),$$

$$f(-3), f(-2), \cdots, f(n-3),$$

we see that $f(-1) = a - d$, $f(-2) = a - 2d$, and $f(-3) = a - 3d$. Continuing this way, we see that $f(m) = a + md$ for all $m \leq 0$. This shows that f is linear.

In conclusion, we have shown the following:

- If $n \geq 3$ is prime, there exists a nonlinear function f mapping n-term arithmetic progressions to n-term arithmetic progressions.
- If $n \geq 3$ is composite and there exists an m such that $f(m) = f(m+n)$, then any function mapping n-term arithmetic progressions to n-term arithmetic progressions must be constant and hence linear.
- If $n \geq 3$ is composite and $f(m) \neq f(m+n)$ for every m, then any function f mapping n-term arithmetic progressions to n-term arithmetic progressions must map any $n+1$ consecutive integers to an arithmetic sequence, with the first and last of these integers mapping to the first and last terms of the sequence in some order. This, in turn, implies that f is a linear function.

This shows that an integer $n \geq 3$ has the desired property if and only if it is composite.