USA Mathematical Talent Search
Round 2 Solutions
Year 32 — Academic Year 2020–2021
www.usamts.org

1/2/32. In the grid below, fill each gray cell with one of the numbers from the provided bank, with each number used once, and fill each white cell with a positive one-digit number. The number in a gray cell must equal the sum of the numbers in all touching *white* cells, where two cells sharing a vertex are considered touching. All of the terms in each of these sums must be distinct, meaning that two white cells with the same digit may not touch the same gray cell.

	5				8	
4						
	3					1
					7	
	8			1		4
					2	

Bank: 15, 23, 28, 35, 36, 38, 40, 42, 44

There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the constraints above. (Note: in any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

Solution

8	5	1	15	5	8	9
4	38	2	3	4	35	2
6	3	9	44	6	28	1
4	40	7	8	5	7	3
1	8	2	42	1	36	4
3	23	9	4	6	2	8

2/2/32. Infinitely many math beasts stand in a line, all six feet apart, wearing masks, and with clean hands. Grogg starts at the front of the line, holding n pieces of candy, $n \geq 1$, and everyone else has none. He passes his candy to the beasts behind him, one piece each to the next n beasts in line. Then, Grogg leaves the line. The other beasts repeat this process: the beast in front, who has k pieces of candy, passes one piece each to the next k beasts in line, and then leaves the line. For some values of n, another beast, besides Grogg, temporarily holds all the candy. For which values of n does this occur?

Solution

We present two different solutions to this problem.

Solution 1.

We claim that the only values of n where some other beast besides Grogg temporarily holds all the candy are $n = 1$ and $n = 2$. If $n = 1$, then Grogg will pass the only piece of candy to the beast behind him, and the condition is satisfied. If $n = 2$, then Grogg will pass one piece of candy to each of the two beasts behind him. Then, the beast who was initially behind Grogg will pass their piece of candy to the beast behind them, meaning that the beast who was originally two places behind Grogg will be holding both pieces of candy. So this indeed occurs when $n = 1$ or $n = 2$.

To finish the proof, we claim that when $n \geq 3$, no other beast besides Grogg will teporarily hold all the candy. Let $n \geq 3$ be a fixed integer. Suppose that each beast in line is numbered for their initial place in line, so that Grogg is given the number 1, the beast behind Grogg is given the number 2, and so on. Now, we prove a lemma:

Lemma. When each beast numbered $s \geq 3$ reaches the front of the line, they will have at least two pieces of candy, and the beast behind them will have at least one piece of candy.

Proof. We prove this by induction on s: First, we look at the base case $s = 3$. After Grogg passes his candy and leaves the line, each of the beasts numbered 2 through $n + 1$ will have one piece of candy. After the beast numbered 2 passes their candy and leaves the line, the beast numbered 3 will have two pieces of candy, and each of the beasts numbered 4 through $n + 1$ will each have one piece of candy. Since $n \geq 3$, the claim is true for $s = 3$.

Suppose that when the beast numbered $s = k$ reaches the front of the line, they have at least two pieces of candy, and the beast numbered $k+1$ has at least one piece of candy. Then as the beast numbered k passes their candy, they must give one piece to the beast numbered $k + 1$ and one piece to the beast numbered $k + 2$. Therefore, when the beast numbered k leaves the line and the beast numbered $k+1$ reaches the front of the line, the beast numbered $k + 1$ will have at least two pieces of candy and the beast numbered $k + 2$ will have at least one piece of candy, and induction is complete. $\square$

Our Lemma allows us to finish the proof as follows: As described in the base case of the Lemma, when the beast numbered 2 reaches the front of the line, they will only have 1 piece of candy, which is not all the candy as $n \geq 3$. Then, when every other beast reaches the front of the line, they will not have all the candy as the beast behind them will have at least one piece of candy. Furthermore, one can see that as each beast passes out their candy in turn, they will pass their candy to at least two other beasts, so no beast will temporarily hold all the candy during the candy-passing process. Therefore, as our choice of n was arbitrary, when $n \geq 3$, no other beast besides Grogg will temporarily hold all the candy.

Solution 2.

We claim that the only values of n where some other beast besides Grogg temporarily holds all the candy are $n = 1$ and $n = 2$. As in the previous solution, we can directly show this indeed occurs when $n = 1$ or $n = 2$. Now, suppose that there exists some beast besides Grogg, say Lizzie, who at some point temporarily holds all the candy, and that Lizzie is the first beast in line for which this is true. We claim that if this happens, then $n \leq 2$, completing our proof.

Call one iteration of the candy-passing process described in the problem a *candy-passing cycle*, so that after a positive integer number of candy-passing cycles, no beast is in the middle of passing out candy, and the beast who previously passed out candy has left the line. Then Lizzie must hold all the candy after some integer number of candy-passing cycles.

Consider the distribution of candy among the beasts one candy-passing cycle *before* Lizzie first holds all the candy. We make a series of observations about this distribution of candy:

- Lizzie cannot be the first beast in line at this point, or she would not be in the line one candy-passing cycle later, a contradiction.
- Only Lizzie and one other beast (say Winnie) could possibly have any candy: If two or more beasts aside from Lizzie had candy, only one of them could be at the front of the line, so after one candy-passing cycle, there must be at least one beast aside from Lizzie with candy, a contradiction.
- The beast at the front of the line must have at least one piece of candy: If they have no candy, then they would leave the line in the next candy-passing cycle without changing the distribution of candy, so Lizzie must already be holding all the candy, a contradiction.

Since Lizzie cannot be the first beast in line, Winnie must be, and Lizzie must be standing immediately behind Winnie (or the beast immediately behind Winnie will have some candy after one candy-passing cycle, a contradiction).

If Winnie had more than 1 piece of candy, then Winnie passing out candy would cause the beast immediately behind Lizzie to have some candy one candy-passing cycle later, a contradiction. Therefore, one candy-passing cycle before Lizzie has all the candy, Winnie must have exactly 1 piece of candy and Lizzie must have exactly $n-1$ pieces of candy. However, we claim that Winnie must have at least as many pieces of candy as Lizzie has, and demonstrate this by proving a more general lemma:

Lemma. After any whole number of candy-passing cycles, the amount of candy the beasts in line are holding, from front to back, is nonstrictly monotonically decreasing. That is, the number of pieces of candy each beast has is greater than or equal to the number of pieces of candy the beast behind them has.

Proof. We prove this by induction on c, the number of elapsed candy-passing cycles. When $c = 0$, the amount of candy the beasts in line are holding is nonstrictly monotonically decreasing, since Grogg has all the candy.

Suppose that when $c = m$, the amount of candy the beasts in line are holding is nonstrictly monotonically decreasing. In particular, suppose that at this point, the ith beast in line is holding a_i pieces of candy, so the first $n+1$ beasts in line at this point are holding $a_1 \geq a_2 \geq \cdots \geq a_n \geq a_{n+1}$ pieces of candy (noting that at most n beasts in line can hold candy at any given point). Then after the first beast passes their candy out and leaves the line, the first n beasts will then be holding $a_2 + 1, \ldots, a_{a_1+1} + 1, a_{a_1+2}, \ldots, a_{n+1}$ candies respectively. Since $a_2 \geq a_3 \geq \cdots \geq a_n \geq a_{n+1}$, it follows that $a_2 + 1 \geq \cdots \geq a_{a_1+1} + 1 \geq a_{a_1+2} \geq \cdots \geq a_{n+1}$. Therefore, when $c = m+1$, the amount of candy the beasts in line are holding is nonstrictly monotonically decreasing, and induction is complete. $\square$

From our Lemma, since Winnie is holding 1 piece of candy, Lizzie is holding $n-1$ pieces of candy, and Lizzie is standing immediately behind Winnie, then $1 \geq n-1$. But this means that $n \leq 2$, as desired.

3/2/32. Given a nonconstant polynomial with real coefficients $f(x)$, let $S(f)$ denote the sum of its roots. Let p and q be nonconstant polynomials with real coefficients such that $S(p) = 7$, $S(q) = 9$, and $S(p-q) = 11$. Find, with proof, all possible values for $S(p+q)$.

Solution

We claim all possible values for $S(p+q)$ are $3, 7, 8$, and $\dfrac{25}{3}$.

Note that by Vieta's formulas, $S(f)$ is completely determined by the first two coefficients of f, and in particular, if $S(f) = k$, then the degree-d polynomial

$$f(x) = c_d x^d + c_{d-1}x^{d-1} + \cdots + c_1 x + c_0$$

satisfies $k = -\dfrac{c_{d-1}}{c_d}$. Therefore, the polynomials p and q must be of the form

$$\begin{aligned} p(x) &= ax^n - 7ax^{n-1} + \cdots \\ q(x) &= bx^m - 9bx^{m-1} + \cdots \end{aligned}$$

where a, b are nonzero real numbers, n, m are the respective degrees of $p(x)$ and $q(x)$, and terms of lower order are omitted.

Next, we consider how the degrees of the polynomials $p(x)$ and $q(x)$ compare, and claim that $|n - m| \le 1$. If $n \ge m+2$, then the leading terms of the polynomial $p(x) - q(x)$ would be $ax^n - 7ax^{n-1}$, but this is not possible as then $S(p-q) = 7$, not 11. Similarly, if $m \ge n+2$, the leading terms of the polynomial $p(x) - q(x)$ would be $-bx^m + 9bx^{m-1}$, but $S(p-q)$ would then be 9, not 11. Therefore, we have $|n - m| \le 1$.

This means that either $n = m, n = m+1$, or $n = m-1$. We analyze each of these cases separately, and determine the possible values of $S(p+q)$ in each case:

- Suppose $n = m$. Then

  $$p(x) - q(x) = (a-b)x^n - (7a - 9b)x^{n-1} + \cdots,$$

  so either $a = b$ or $\dfrac{7a-9b}{a-b} = 11$.

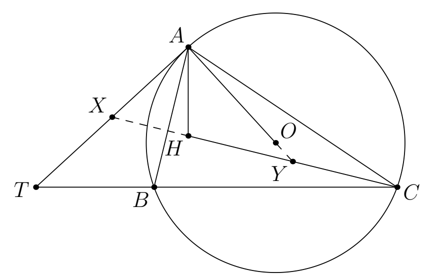
  - If $a = b$, then $p(x) + q(x) = 2ax^n - 16ax^{n-1} + \cdots$, so $S(p+q) = \dfrac{16a}{2a} = 8$. This is feasible, by letting $p(x) = x^2 - 7x$ and $q(x) = x^2 - 9x + 22$.
  - If $\dfrac{7a-9b}{a-b} = 11$, then $b = 2a$, giving $p(x) + q(x) = 3ax^n - 25ax^{n-1} + \cdots$, so $S(p+q) = \dfrac{25a}{3a} = \dfrac{25}{3}$. This is feasible, by letting $p(x) = x - 7$ and $q(x) = 2x - 18$.

- Suppose $n = m + 1$. Then
$$p(x) - q(x) = ax^n - (7a + b)x^{n-1} + \cdots,$$
so $\dfrac{7a + b}{a} = 11$. Then $b = 4a$, so $p(x)+q(x) = ax^n-3ax^{n-1}+\cdots$ and $S(p+q) = \dfrac{3a}{a} = 3$. This is feasible, by letting $p(x) = x^2 - 7x$ and $q(x) = 4x - 36$.

- Suppose $n = m - 1$. Then
$$p(x) - q(x) = -bx^{n+1} - (-a - 9b)x^n + \ldots,$$
so $\dfrac{-a - 9b}{-b} = 11$. Then $a = 2b$, so $p(x) + q(x) = bx^{n+1} - 7bx^n + \ldots$ and $S(p + q) = \dfrac{7b}{b} = 7$. This is feasible, by letting $p(x) = 2x - 14$ and $q(x) = x^2 - 9x$.

Thus, there are four possible values for $S(p + q)$: $\boxed{3, 7, 8, \text{and } \dfrac{25}{3}}$.

4/2/32. Let ABC be a triangle with $AB < AC$. As shown below, T is the point on $\overline{BC}$ such that $\overline{AT}$ is tangent to the circumcircle of $\triangle ABC$. Additionally, H and O are the orthocenter and circumcenter of $\triangle ABC$, respectively. Suppose that $\overline{CH}$ passes through the midpoint of $\overline{AT}$. Prove that $\overline{AO}$ bisects $\overline{CH}$.



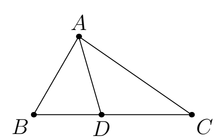
Solution

We present two different solutions to this problem.

Solution 1.

We claim that $\dfrac{AX}{XT} = \dfrac{HY}{YC}$, implying the result. Before approaching the actual problem, we state and prove a useful lemma (colloquially known as the *Ratio Lemma*):

Ratio Lemma. Let D be a point on side BC of $\triangle ABC$. Then $\dfrac{BD}{DC} = \dfrac{\sin\angle BAD}{\sin\angle CAD} \cdot \dfrac{AB}{AC}$.



Proof. By the Law of Sines on $\triangle ABD$ and $\triangle ADC$, we have

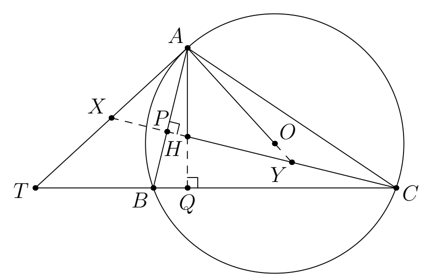
$$\frac{BD}{\sin\angle BAD} = \frac{AB}{\sin\angle ADB} \quad \text{and} \quad \frac{DC}{\sin\angle CAD} = \frac{AC}{\sin\angle ADC}.$$

Since $\angle ADB$ and $\angle ADC$ are supplementary, it follows that $\sin\angle ADB = \sin\angle ADC$. Therefore, by rearranging, we see that

$$\frac{AB}{BD} \cdot \sin\angle BAD = \sin\angle ADB = \sin\angle ADC = \frac{AC}{DC} \cdot \sin\angle CAD.$$

Therefore, $\dfrac{AB}{BD} \cdot \sin\angle BAD = \dfrac{AC}{DC} \cdot \sin\angle CAD$, and rearranging this equation gives us $\dfrac{BD}{DC} = \dfrac{\sin\angle BAD}{\sin\angle CAD} \cdot \dfrac{AB}{AC}$, as desired. $\square$

As noted in the diagram below, let X be the intersection of the lines $\overline{CH}$ and $\overline{AT}$, and let Y be the intersection of the lines $\overline{CH}$ and $\overline{AO}$. Let P be the intersection of $\overline{AB}$ and $\overline{CX}$, and Q be the intersection of the extension of $\overline{AH}$ with $\overline{BC}$. Note that $\overline{CP}$ and $\overline{AQ}$ are altitudes, so it follows that $\triangle APC, \triangle BPC, \triangle AQC$, and $\triangle AQB$ are right:



Applying the Ratio Lemma to $\triangle ACT$ with point X on AT, we get

$$\frac{AX}{XT} = \frac{\sin \angle ACX}{\sin \angle TCX} \cdot \frac{AC}{CT}.$$

Since $\triangle APC$ is right, we have

$$\sin \angle ACX = \sin \angle ACP = \cos \angle PAC = \cos \angle BAC.$$

Similarly, since $\triangle BPC$ is right,

$$\sin \angle TCX = \sin \angle BCP = \cos \angle CBP = \cos \angle CBA.$$

Applying the Law of Sines to $\triangle ACT$, we find $\dfrac{AC}{CT} = \dfrac{\sin \angle CTA}{\sin \angle TAC}$. By looking at inscribed angles in the circumcircle of $\triangle ABC$, we see that

$$\angle CTA = \frac{1}{2}\left(\widehat{AC} - \widehat{AB}\right) = \angle CBA - \angle ACB$$

and

$$\angle TAC = \frac{1}{2}\left(\widehat{BC} + \widehat{AB}\right) = \angle BAC + \angle ACB.$$

Substituting all of this into our original expression for $\dfrac{AX}{XT}$, we get

$$\frac{AX}{XT} = \frac{\sin \angle ACX}{\sin \angle TCX} \cdot \frac{AC}{CT} = \frac{\cos \angle BAC}{\cos \angle CBA} \cdot \frac{\sin(\angle CBA - \angle ACB)}{\sin(\angle BAC + \angle ACB)}.$$

Finally, we undertake a similar line of reasoning in $\triangle HAC$ to obtain an expression for $\dfrac{HY}{YC}$ identical to that in our equality chain above. Applying the Ratio Lemma in $\triangle HAC$ with point Y on HC, we get

$$\frac{HY}{YC} = \frac{\sin \angle HAY}{\sin \angle CAY} \cdot \frac{AH}{AC}.$$

Furthermore, by the Law of Sines on $\triangle HAC$, we have $\dfrac{AH}{AC} = \dfrac{\sin\angle ACH}{\sin\angle CHA}$, so

$$\frac{HY}{YC} = \frac{\sin\angle HAY}{\sin\angle CAY} \cdot \frac{\sin\angle ACH}{\sin\angle CHA}.$$

Since $\triangle APC$ is right, we have

$$\sin\angle ACH = \sin\angle ACP = \cos\angle PAC = \cos\angle BAC.$$

Since $\overline{AT}$ is tangent to the circumcircle of $\triangle ABC$, it follows from our work above (and since $\triangle BPC$ is right) that

$$\begin{aligned}
\sin\angle CAY &= \sin(\angle TAC - \angle TAY) \\
&= \sin(\angle TAC - \angle TAO) \\
&= \sin(\angle BAC + \angle ACB - 90^\circ) \\
&= \sin(90^\circ - \angle CBA) \\
&= \cos\angle CBA.
\end{aligned}$$

Since $\overline{AT}$ is a tangent and $\overline{AH}$ is an altitude, we have $\angle HAY = \angle TAY - \angle TAH = 90^\circ - \angle TAH = \angle CTA$. By our work above, $\angle CTA = \angle CBA - \angle ACB$, so $\sin\angle HAY = \sin(\angle CBA - \angle ACB)$. Then by looking at the right triangles $\triangle AQC$ and $\triangle APC$, we have

$$\begin{aligned}
\sin\angle CHA &= \sin(180^\circ - \angle HAC - \angle ACH) \\
&= \sin((90^\circ - \angle HAC) + (90^\circ - \angle ACH)) \\
&= \sin(\angle ACB + \angle BAC).
\end{aligned}$$

Substituting all this into our original expression, we get

$$\begin{aligned}
\frac{HY}{YC} &= \frac{\sin\angle HAY}{\sin\angle CAY} \cdot \frac{\sin\angle ACH}{\sin\angle CHA} \\
&= \frac{\sin(\angle CBA - \angle ACB)}{\cos\angle CBA} \cdot \frac{\cos\angle BAC}{\sin(\angle BAC + \angle ACB)} \\
&= \frac{\cos\angle BAC}{\cos\angle CBA} \cdot \frac{\sin(\angle CBA - \angle ACB)}{\sin(\angle BAC + \angle ACB)}.
\end{aligned}$$

This is equal to the expression we found for $\dfrac{AX}{XT}$, which means that $\dfrac{AX}{XT} = \dfrac{HY}{YC}$. Since $\overline{CH}$ passes through point X, which we are supposing to be the midpoint of $\overline{AT}$, we have $AX = XT$. It follows that $HY = YC$, and therefore $\overline{AO}$ bisects $\overline{CH}$, as desired.

Solution 2.

For a primer on complex bashes, see `https://web.evanchen.cc/handouts/cmplx/en-cmplx.pdf`.

In this solution, lowercase letters denote complex numbers and WLOG a, b, and c all lie on the unit circle. This means that $\overline{a} = 1/a$, $\overline{b} = 1/b$, and $\overline{c} = 1/c$.

The equation for a line in the complex plane passing through points α and β on the unit circle is $z + \alpha\beta\overline{z} = \alpha + \beta$. Setting $\alpha = \beta$ gives the equation for a line tangent to the unit circle. The tangent line through a has equation $z + a^2\overline{z} = 2a$, and the line through b and c has equation $z + bc\overline{z} = b + c$. Since t is the intersection of these lines, we have:

$$\begin{aligned} t + a^2\overline{t} &= 2a, \\ t + bc\overline{t} &= b + c. \end{aligned}$$

Isolating t on both sides:

$$\begin{aligned} t &= 2a - a^2\overline{t}, \\ t &= b + c - bc\overline{t}. \end{aligned}$$

Setting both right-hand sides equal to each other and solving:

$$\overline{t} = \frac{2a - b - c}{a^2 - bc}.$$

Undoing the conjugate:

$$t = \overline{\left(\frac{2a - b - c}{a^2 - bc}\right)} = \frac{2/a - 1/b - 1/c}{1/a^2 - 1/(bc)} = \frac{a^2b + a^2c - 2abc}{a^2 - bc}.$$

Now let's find the equations for h, o, and x. It is well-known that $h = a + b + c$ and $o = 0$. Since x is the midpoint of a and t,

$$x = \frac{a + t}{2}.$$

Let y be the midpoint of c and h. Our goal now is to prove that a, o, and y are collinear.

$$y = \frac{c + h}{2} = \frac{a + b + 2c}{2}$$

Since c, x, and h are collinear, the quantity $\frac{x-c}{h-c}$ must be real:

$$\begin{aligned} \frac{x - c}{h - c} &= \frac{\frac{a+t}{2} - c}{a + b + c - c} \\ &= \frac{a + t - 2c}{2(a + b)}. \end{aligned}$$

Since this quantity is real:

$$\begin{aligned}
\frac{a+t-2c}{2(a+b)} &= \overline{\left(\frac{a+t-2c}{2(a+b)}\right)}\\
\frac{a+t-2c}{2(a+b)} &= \frac{1/a+\bar{t}-2/c}{2/a+2/b}\\
a+t-2c &= b+ab\bar{t}-2ab/c\\
a+\frac{a^2b+a^2c-2abc}{a^2-bc}-2c &= b+\frac{ab(2a-b-c)}{a^2-bc}-2ab/c\\
0 &= \frac{(a-c)(2bc^2+cb^2-ca^2-2ba^2)}{c(a^2-bc)}.
\end{aligned}$$

Since $\frac{a-c}{c(a^2-bc)}$ is not equal to 0, we can divide it out, getting:

$$2bc^2+cb^2=ca^2+2ba^2.$$

Continuing the algebra:

$$\begin{aligned}
bc(b+2c) &= a^2(c+2b)\\
\frac{b+2c}{a} &= \frac{a(c+2b)}{bc}\\
\frac{b+2c}{a} &= \frac{1/b+2/c}{1/a}\\
\frac{b+2c}{a} &= \overline{\left(\frac{b+2c}{a}\right)}.
\end{aligned}$$

So $\frac{b+2c}{a}$ is real. Thus, $\frac{1}{2}\times\frac{b+2c}{a}+\frac{1}{4}=\frac{a+b+2c}{2a}=\frac{y-o}{a-o}$ is also real, so a, o, and y must be collinear. $\square$

5/2/32. Let a_1 be any positive integer. For all i, write 5^{2020} times a_i in base 10, replace each digit with its remainder when divided by 2, read off the result in binary, and call that a_{i+1}. Prove that $a_N = a_{N+2^{2020}}$ for all sufficiently large N.

Solution

We prove a stronger result: For any fixed positive integer k, if we replace every instance of 2020 in the problem with k, then the result will be true. Let $f(n)$ be the result of taking the digits of $5^k \cdot n \pmod 2$ and reading them in binary. (That is, $f(a_i) = a_{i+1}$ for all i.) To show this, we prove a series of lemmas:

Lemma 1. For all n, b with $b \le k$,

(a) $f(n + 2^b) \equiv f(n) + 2^b \pmod{2^{b+1}}$, and

(b) $f(n + 2^{b+1}) \equiv f(n) \pmod{2^{b+1}}$.

Proof. For part (a), since $b \le k$, we have

$$5^k \cdot (n + 2^b) = 5^k \cdot n + 10^b \cdot 5^{k-b}.$$

Notice that $10^b \cdot 5^{k-b}$ ends in b zeroes preceded by either a 1 or 5. Therefore, when it is added to $5^k \cdot n$, the parities of the last b digits do not change while the parity of the one with place value 10^b does. When we convert all digits to 0 or 1 based on their parity, then between $f(n + 2^b)$ and $f(n)$, only the digit in place value 2^b changes out of the last $b + 1$ binary digits. In modular arithmetic, this is exactly the given statement.

Similarly, for part (b), we have

$$5^k \cdot (n + 2^{b+1}) = 5^k \cdot n + 10^b \cdot 5^{k-b} \cdot 2.$$

Notice that $10^b \cdot 5^{k-b}$ ends either in b zeroes preceded with a 2, or with $b+1$ zeroes. Regardless, when it is added to $5^k \cdot n$, the parities of the last $b+1$ digits do not change, so when converted to binary, both $f(n + 2^{b+1})$ and $f(n)$ have the same last $b + 1$ binary digits, as desired. □

Note that Lemma 1(b) implies that f is a well-defined function when its input and output are taken modulo any power of 2 that is at most 2^{k+1}. (That is, if two inputs to f are congruent modulo that power of 2, their outputs will be as well.)

Lemma 2. For all i, b with $b \le k$, $a_i \equiv a_{i+2^b} \pmod{2^{b+1}}$.

Proof. We induct on b. For the base case $b = 0$, we are working mod 2, so we only need to look at the cases where $a_i = 0$ or $a_i = 1$. If $a_i = 0$, then clearly $a_{i+1} = 5^k \cdot 0 \equiv 0 \pmod 2$. Also, if $a_i = 1$, then $a_{i+1} = 5^k \cdot 1 \equiv 5^k \equiv 1 \pmod 2$. By Lemma 1(b), this establishes the base case $b = 0$.

Suppose the given statement is true for $b-1$. Since $a_1 \equiv a_{1+2^{b-1}} \pmod{2^b}$, there are two cases for the relationship between these two mod 2^{b+1}:

Case 1: $a_1 \equiv a_{1+2^{b-1}} \pmod{2^{b+1}}$.

In this case, since f is well-defined modulo 2^{b+1}, we have $a_i \equiv a_{i+2^{b-1}} \pmod{2^{b+1}}$ for all i, which means $a_i \equiv a_{i+2^{b-1}} \equiv a_{i+2^b} \pmod{2^{b+1}}$ for all i. This completes the induction for this case.

Case 2: $a_1 \equiv a_{1+2^{b-1}} + 2^b \pmod{2^{b+1}}$.

In this case, by Lemma 1(a) and the well-definedness of f, we have $a_i \equiv a_{i+2^{b-1}} + 2^b \pmod{2^{b+1}}$ for all i, which implies $a_i \equiv a_{i+2^{b-1}} + 2^b \equiv a_{i+2^b} + 2 \cdot 2^b \equiv a_{i+2^b} \pmod{2^{b+1}}$ once again. This completes the induction for this case. $\square$

Lemma 3. For all n, $f(n) < \max(n, 2^{k+1})$.

Proof. Let n be a fixed positive integer, and let d be the unique integer such that

$$2^k \cdot 10^d \le n < 2^k \cdot 10^{d+1}.$$

Then $n \cdot 5^k < 10^{d+k+1}$. In particular, $n \cdot 5^k$ has at most $d+k+1$ digits, meaning $f(n)$ is less than 2^{d+k+1}. If $d = 0$, then $f(n) < 2^{k+1}$ and we are done. Otherwise, we have $d \ge 1$, in which case we have

$$f(n) < 2^{d+k+1} < 2^k \cdot 10^d \le n,$$

so $f(n) < n$. Since our choice of n was arbitrary, we are done. $\square$

Now we finish the problem. Lemma 3 tells us that as f is iteratively applied to a_1, the result keeps getting smaller until it is at some point less than 2^{k+1}, so there exists an N such that $0 \le a_i < 2^{k+1}$ for all $i \ge N$. Then for all $i \ge N$, setting $b = k$ in Lemma 2 tells us that $a_i = a_{i+2^k}$, and the proof is complete.

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