USA Mathematical Talent Search

Round 1 Solutions

Year 32 — Academic Year 2020–2021

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1/1/32. Fill in each empty cell of the grid with a digit from 1 to 8 so that every row and every column contains each of these digits exactly once. Some diagonally adjacent cells have been joined together. For these pairs of joined cells, the same number must be written in both.

There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

1							
2	3						
	4	5					
		6		1			
			7		1		
					3	4	
						2	8
							5

Solution

1	7	3	8	6	4	5	2
2	3	7	6	4	5	8	1
8	4	5	2	7	6	1	3
4	8	6	5	1	2	3	7
3	2	8	7	5	1	6	4
7	5	2	1	8	3	4	6
5	6	1	4	3	7	2	8
6	1	4	3	2	8	7	5

2/1/32. Is it possible to fill in a 2020×2020 grid with the integers from 1 to 4,080,400 so that the sum of each row is 1 greater than the previous row?

Solution

We claim such a grid is not possible, and show this by contradiction. Suppose this were possible, and fix an arrangement of the integers from 1 through 4,080,400 in a 2020×2020 grid where the sum of each row is 1 greater than the previous row. Let S be the sum of the entries in the first row of this grid. Note that since each entry in the grid is an integer, S must also be an integer. Since the sum of each row is 1 greater than the previous row, it follows that the sum of the entries in the second row of the grid is $S+1$, the sum of the entries in the third row of the grid is $S+2$, and so on, so that the sum of the entries in the 2020^{th} row of the grid is $S+2019$. Since the entries in the grid are the integers from 1 to 4,080,400, it then follows that

$$S+(S+1)+(S+2)+\cdots+(S+2018)+(S+2019) = 1+2+3+\cdots+4{,}080{,}399+4{,}080{,}400.$$

Since the sum of the first n positive integers is $\dfrac{n(n+1)}{2}$, the left-hand side of this equation is equal to

$$2020S + \frac{2019 \cdot 2020}{2} = 1010(2S+2019).$$

Furthermore, since $4{,}080{,}400 = 2020^2$, the right-hand side of this equation is equal to

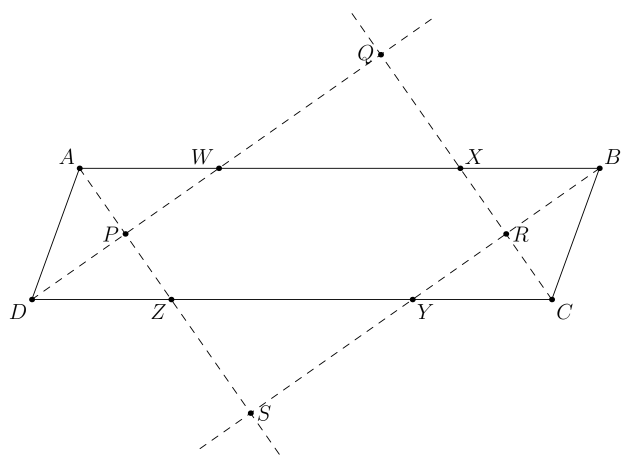
$$\frac{2020^2(2020^2+1)}{2} = 1010 \cdot 2020 \cdot (2020^2+1).$$

Observe that the right-hand side of this equation is divisible by 4. However, the left-hand side of this equation is not divisible by 4, since $2S+2019$ is odd and 1010 is not divisible by 4. This is a contradiction, so it follows that it is not possible to fill in this grid in the desired manner. $\square$

3/1/32. The bisectors of the internal angles of parallelogram $ABCD$ determine a quadrilateral with the same area as $ABCD$. Given that $AB > BC$, compute, with proof, the ratio AB/BC.

Solution

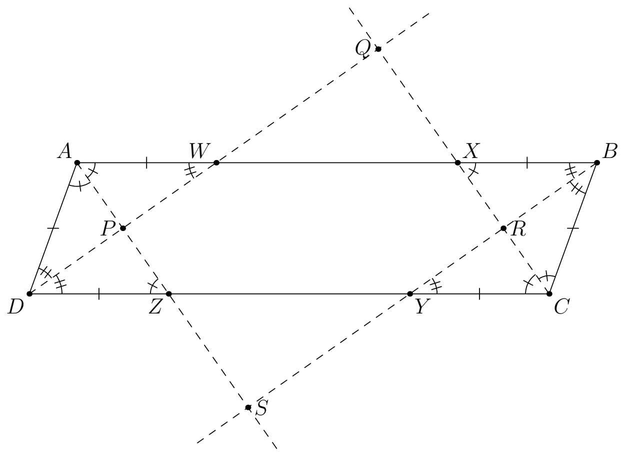
We claim that $\dfrac{AB}{BC} = 2 + \sqrt{3}$. Let $PQRS$ be the quadrilateral determined by the angle bisectors, and let W, X, Y, Z be the points where the bisectors intersect $\overline{AB}$ and $\overline{CD}$, as shown in the following diagram:



First, we claim that $AW = XB = CY = DZ = AD = BC$. Because $AB \parallel CD$ and AS bisects $\angle DAW$, it follows that $\angle AZD = \angle WAZ = \angle ZAD$. Similarly, $\angle AWD = \angle WDZ = \angle ADW$. Since $\triangle ADP$ and $\triangle ZDP$ have side DP in common, it follows that they are congruent, and similarly, $\triangle ADP \cong \triangle AWP$. By triangle congruence, it follows that $AW = AD = DZ$.

Similarly, we can show that $\triangle CYR \cong \triangle CBR \cong \triangle XBR$, so $XB = BC = CY$. Finally, since $ABCD$ is a parallelogram, we know $AD = BC$, demonstrating the desired equality. Also, note that since $AD = BC$, we actually have

$$\triangle AWP \cong \triangle XBR \cong \triangle CYR \cong \triangle ZDP \cong \triangle ADP \cong \triangle CBR.$$



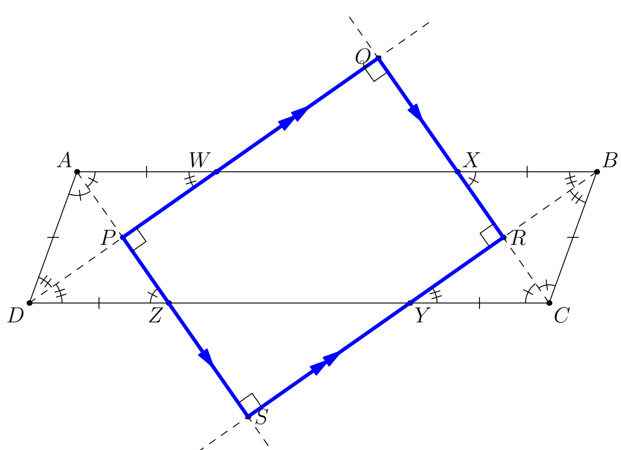
Next, we claim that $\triangle ZYS \cong \triangle XWQ$. By the properties of parallelograms and angle bisectors, we see that

$$\angle QDC = \frac{1}{2}\angle ADC = \frac{1}{2}\angle ABC = \angle ABS = \angle BYC,$$

so $DQ \parallel BS$. Similarly, $AS \parallel CQ$. Then,

$$\angle DAP + \angle ADP = \frac{1}{2}\left(\angle DAB + \angle ADC\right) = \frac{1}{2} \cdot 180^\circ = 90^\circ,$$

so $\angle APD$ and $\angle QPS$ are right. It follows from the properties of parallel lines that $PQRS$ is a rectangle.



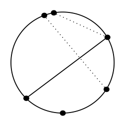
Now, by vertical angles, we have $\angle QWX = \angle AWP = \angle CYR = \angle SYZ$, and since $AW = XB = CY = ZD$, we also have $WX = ZY = AB - 2 \cdot BC$. Because $\triangle ZYS$ and $\triangle XWQ$ are right triangles with identical angles and the same hypotenuse length, we have $\triangle ZYS \cong \triangle XWQ$.

Finally, we show $\dfrac{AB}{BC} = 2+\sqrt{3}$. Because hexagon $WXRYZP$ is contained in both $ABCD$ and $PQRS$ - which have the same area - and by the triangle congruences noted above, we have

$$2[\triangle ZYS] = 6[\triangle DZP] \text{ or } [\triangle ZYS] = 3[\triangle DZP].$$

Since $\angle DPZ$ and $\angle ZSY$ are right and $\angle PZD = \angle SZY$ by vertical angles, it also follows that $\triangle ZYS$ and $\triangle DZP$ are similar. Thus, $ZY = \sqrt{3} \cdot DZ$. Finally, since $DZ = BC$ and $ZY = AB - 2 \cdot BC$, we have $\sqrt{3} \cdot BC = AB - 2 \cdot BC$ and hence $\dfrac{AB}{BC} = \boxed{2 + \sqrt{3}}$. $\square$

4/1/32. Two beasts, Rosencrans and Gildenstern, play a game. They have a circle with n points ($n \geq 5$) on it. On their turn, each beast (starting with Rosencrans) draws a chord between a pair of points in such a way that **any two** chords have a shared point. (The chords either intersect or have a common endpoint.) For example, two potential legal moves for the second player are drawn below with dotted lines.



The game ends when a player cannot draw a chord. The last beast to draw a chord wins. For which n does Rosencrans win?

Solution

For this problem, it suffices to demonstrate a winning strategy for Rosencrans for all n where Rosencrans wins, and to demonstrate a winning strategy for Gildenstern for all other n. However, in this solution, we make a stronger claim: this game will *always* end after n moves, no matter what moves Rosencrans and Gildenstern make. Consequently, Rosencrans will win for all odd positive integers $n \geq 5$.

Starting with an arbitrary point, label the points on the circle with the integers $0, 1, 2, \ldots, n-1$ in clockwise order, and denote by $\overline{s,t}$ the chord with endpoints labeled s and t. First, we consider when two chords will have a shared point. Suppose $\overline{s,t}$ and $\overline{u,v}$ are two chords with four distinct endpoints (if two chords have a shared endpoint, these two chords will certainly have a shared point), and without loss of generality, suppose that s has the smallest numerical label of the four endpoints. Then the two chords will intersect if and only if $\overline{u,v}$ has one endpoint on each of the clockwise-directed arcs from the point labeled s to the point labeled t and from the point labeled t to the point labeled s respectively - that is, if $s < u < t < v$ or $s < v < t < u$. Hence, more generally, two chords will intersect if and only if $s \leq u \leq t \leq v$ or $s \leq v \leq t \leq u$.

For all integers $0 \leq k \leq n-1$, define the k^{th} *parallel class* of chords as the set of all chords whose endpoint labels sum to $k \pmod n$. (This terminology is motivated by the fact that all the chords in a given parallel class are indeed parallel when all the points on our circle are evenly spaced out.) Note that by construction, this partitions the set of chords into n subsets. Now, we make two claims about parallel classes:

Lemma 1. No two chords in the same parallel class have a shared point.

Lemma 2. Let $0 \leq k \leq n-1$ be an integer. If, at any point in the game, a chord in the k^{th} parallel class has not been drawn, the player whose turn it is can legally draw a chord from the k^{th} parallel class.

These two lemmas allow us to finish the proof, as follows. Since any two chords drawn by Rosencrans and Gildenstern must have a shared point, Lemma 1 tells us that they can't draw two chords from the same parallel class over the course of the game. Thus, during the game, they can draw at most one chord from each parallel class, and since there are n parallel classes, the game will end in at most n moves.

Then, as long as Rosencrans and Gildenstern have collectively made fewer than n moves, Lemma 2 tells us that the player whose turn it is will always be able to make a legal move. Hence, the game will always end in exactly n moves.

We finish by proving both of these lemmas:

Proof of Lemma 1. Firstly, no two chords in the same parallel class can have a shared endpoint, since a chord in the k^{th} parallel class with endpoint s must, by construction, also have endpoint $k - s \pmod{n}$. Now, suppose $\overline{s,t}$ and $\overline{u,v}$ are both in the k^{th} parallel class, and without loss of generality, suppose s has the smallest numerical label of the four endpoints. It is impossible for $s < u < t < v$. Suppose this were the case. We know that $s + t \equiv u + v \pmod{n}$, but since $s < u$ and $t < v$, it must be the case that $s + t < u + v$. Also, $u \le t - 1$ and $v \le s + n - 1$. So $s + t < u + v \le s + t + (n - 2) < s + t + n$. This is a contradiction, so it can't be the case that $s < u < t < v$. Similarly, it is impossible for $s < v < t < u$, and so by our previous work, no two chords in the same parallel class have a shared point. $\square$

Proof of Lemma 2. If no chords have been drawn, clearly Rosencrans can draw in any chord in the k^{th} parallel class. If only one chord $\overline{s,t}$ has been drawn (which is not in the k^{th} parallel class), then Gildenstern can attempt to pick an endpoint of this chord and draw a chord in the k^{th} parallel class containing this endpoint. If this is not possible, this means that $s \equiv k - s \pmod{n}$ and $t \equiv k - t \pmod{n}$, so $2s \equiv 2t \equiv k \pmod{n}$. In this case, Gildenstern can draw in the chord between the endpoints with labels $s + 1 \pmod{n}$ and $s - 1 \pmod{n}$. Hence, we assume that at least two chords (from distinct parallel classes, as shown by Lemma 1) have already been drawn in the game.

Now, let $0 \le i \le n - 1$ be an integer such that the chord $\overline{s,t}$ in the i^{th} parallel class has been drawn and no chords in the parallel classes numbered $i + 1, i + 2, \ldots, k - 1 \pmod{n}$ have been drawn. (This is guaranteed to exist; one can think of "counting backwards from $k \mod n$" until reaching the first number whose parallel class already has a drawn chord.) Similarly, let $0 \le j \le n-1$ be an integer such that the chord $\overline{u,v}$ in the j^{th} parallel class has been drawn and no chords in the parallel classes numbered $k + 1, k + 2, \ldots, j - 1 \pmod{n}$ have been drawn.

Since these chords were drawn legally, the chords $\overline{s,t}$ and $\overline{u,v}$ must either have a common endpoint or intersect inside the circle. Without loss of generality, suppose that $s \le u \le t \le v$. (Note that at most two of these labels are equal.)

Now, we claim it is possible to choose some point labeled w amongst the points labeled $s, s+1, \ldots, u$ and some point labeled x amongst the points labeled $t, t+1, \ldots, v$ such that $\overline{w,x}$ is in the k^{th} parallel class. Consider the sequence of chords

$$\overline{s,t},\ \overline{s+1,t},\ \ldots,\ \overline{u,t},\ \overline{u,t+1},\ \ldots,\ \overline{u,v}.$$

By removing identical chords from the sequence, suppose these are distinct. Then, these chords are in the parallel classes numbered $i, i+1, \ldots, j \pmod n$ respectively, so by construction, one of these chords must be in the k^{th} parallel class.

To complete the proof of the Lemma, we claim that this chord $\overline{w,x}$ can be legally drawn. First, $\overline{w,x}$ has a shared point with both $\overline{s,t}$ and $\overline{u,v}$, since by construction, we have

$$s \le w \le u \le t \le x \le v.$$

In particular, this means that $s \le w \le t \le x$ and $w \le u \le x \le v$, so $\overline{w,x}$ has a shared point with both $\overline{s,t}$ and $\overline{u,v}$.

Now, let $\overline{y,z}$ be any other chord which has already been drawn in the game. We claim that $\overline{w,x}$ and $\overline{y,z}$ must have a shared point. Since $\overline{s,t}$ and $\overline{y,z}$ must have a shared point, without loss of generality suppose $s \le y \le t \le z$. (The other three cases - namely $s \le z \le t \le y, z \le s \le y \le t, y \le s \le z \le t$ - are similar by symmetry and/or relabeling by adding the same number to all point labels, then considering the labels mod n.) Since $\overline{y,z}$ must also have a shared point with $\overline{u,v}$, this leaves us with two cases:

Case 1: $u \le y \le v \le z$. In this case, we have $u \le y \le t$ and $v \le z$, telling us that

$$s \le w \le u \le y \le t \le x \le v \le z.$$

In particular, this means that $w \le y \le x \le z$, and so $\overline{w,x}$ and $\overline{y,z}$ have a shared point.

Case 2: $y \le u \le z \le v$. We claim this case cannot occur. Suppose for a contradiction that this were the case. Then we have $s \le y \le u$ and $t \le z \le v$. Now, consider the sequence of chords

$$\overline{s,t},\ \overline{s+1,t},\ \ldots,\ \overline{y,t},\ \overline{y,t+1},\ \ldots,\ \overline{y,z},\ \overline{y+1,z},\ \ldots,\ \overline{u,z},\ \overline{u,z+1},\ \ldots,\ \overline{u,v}.$$

By removing identical chords from the sequence, suppose these are distinct. Then, these chords are in the parallel classes numbered $i, i+1, \ldots, j \pmod n$ respectively, and in particular, $\overline{y,z}$ must belong to one of these parallel classes. However, by construction, there have only been three chords drawn which belong to one of these parallel classes - $\overline{s,t}$, $\overline{u,v}$, and $\overline{w,x}$ - and by Lemma 1, Rosencrans and Gildenstern cannot draw in two chords from the same parallel class over the course of a game. Hence, this is a contradiction, and this case cannot happen.

This tells us that $\overline{w,x}$ and $\overline{y,z}$ must have a shared point. Since our choice of $\overline{y,z}$ was arbitrary, $\overline{w,x}$ can be legally drawn. $\square$

This concludes the proof of both lemmas. As detailed above, this means that the game will always end in exactly n moves, no matter what moves Rosencrans and Gildenstern make. Hence, Rosencrans will win for all odd $n \geq 5$, as desired. $\square$

5/1/32. Find all pairs of rational numbers (a, b) such that $0 < a < b$ and $a^a = b^b$.

Solution

We claim that all pairs of the form $(a, b) = \left(\left(\frac{n}{n+1}\right)^{n+1}, \left(\frac{n}{n+1}\right)^{n}\right)$, where n is a positive integer, satisfy $0 < a < b$ and $a^a = b^b$, and that these are the only pairs which satisfy these conditions.

First, we show that all such pairs satisfy the given conditions. Fix a positive integer n, and set $a = \left(\frac{n}{n+1}\right)^{n+1}$ and $b = \left(\frac{n}{n+1}\right)^{n}$. Since $0 < \frac{n}{n+1} < 1$, it follows that $0 < a < b$. Also, we have

$$a^a = \left(\frac{n}{n+1}\right)^{n+1\cdot\left(\frac{n}{n+1}\right)^{n+1}} = \left(\frac{n}{n+1}\right)^{\frac{n^{n+1}}{(n+1)^n}} = \left(\frac{n}{n+1}\right)^{n\cdot\left(\frac{n}{n+1}\right)^{n}} = b^b.$$

Thus, all pairs of the form $(a, b) = \left(\left(\frac{n}{n+1}\right)^{n+1}, \left(\frac{n}{n+1}\right)^{n}\right)$ satisfy the given conditions, and it remains to show these are the only pairs which do.

Now, we show that these are the only pairs which satisfy these conditions. Suppose a and b are rational numbers such that $0 < a < b$ and $a^a = b^b$. Then $\frac{b}{a}$ is also a rational number; suppose its reduced form is $\frac{m}{n}$ (so that m, n are positive integers with $\gcd(m, n) = 1$ and $m > n$). Then we can write $b = \frac{m}{n} \cdot a$, and since $a^a = b^b$, we can substitute to get $a^a = \left(\frac{m}{n} \cdot a\right)^{\frac{m}{n}\cdot a}$. From here, we can solve for a. Taking the positive a^{th} root of both sides, we get

$$a = \left(\frac{m}{n} \cdot a\right)^{\frac{m}{n}}.$$

Then, by distributing the right-hand side and dividing by $a^{\frac{m}{n}}$, we get

$$a^{1-\frac{m}{n}} = \left(\frac{m}{n}\right)^{\frac{m}{n}},$$

so

$$a = \left(\frac{m}{n}\right)^{\frac{m}{n-m}}.$$

Note that because $m > n$ and $\gcd(m, n) = 1$, it follows that $n - m$ is a negative integer that has no common factors with m (except 1).

We claim that a is rational if and only if $m-n=1$: First, if $m-n=1$, then $a=\left(\frac{m}{n}\right)^{\frac{m}{n-m}}=\left(\frac{n}{m}\right)^{m}$ is rational. Now, let $m-n=k$, where $k>1$ is a positive integer. We claim that a cannot be rational. Suppose for the sake of contradiction that a is rational. It follows that $a=\left(\frac{m}{n}\right)^{\frac{m}{-k}}=\left(\frac{n^m}{m^m}\right)^{\frac{1}{k}}$. Since $\gcd(m,n)=1$ and a is rational, both m and n are perfect k^{th} powers. But this is impossible: suppose that $m=x^k, n=y^k$, and that $m-n=x^k-y^k=k$. We know that $x>y$ (as $m>n$), and so by the Binomial Theorem we get

$$k=x^k-y^k\geq(y+1)^k-y^k\geq(y^k+ky+1)-y^k=ky+1\geq k+1,$$

which is a contradiction.

Therefore, if a is rational, then $m-n=1$. Since we assumed a is a rational number satisfying our conditions, it then follows that $m-n=1$, and so

$$a=\left(\frac{n}{m}\right)^{m}=\left(\frac{n}{n+1}\right)^{n+1}$$

and

$$b=\frac{m}{n}\cdot a=\frac{n+1}{n}\cdot\left(\frac{n}{n+1}\right)^{n+1}=\left(\frac{n}{n+1}\right)^{n},$$

as desired. □