USA Mathematical Talent Search
Round 3 Solutions
Year 31 — Academic Year 2019–2020
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1/3/31. Fill in each square with a number from 1 to 5; some numbers have been given. If two squares A and B have equal numbers, then A and B cannot share a side, and there also cannot exist a third square C sharing a side with both A and B.

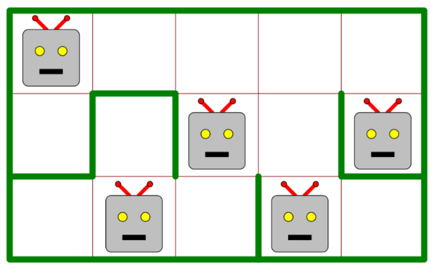
4			2				2	
	1				2			5

There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

Solution

		1	4	5		3	1	4
4	5	3	2	1	4	5	2	3
2	1	4	5	3	2	1	4	5
5	3	2		4	5	3		

2/3/31. An apple orchard's layout is a rectangular grid of unit squares. Some pairs of adjacent squares have a thick wall of grape vines between them. The orchard wants to post some robot sentries to guard its prized apple trees. Each sentry occupies a single square of the layout, and from there it can guard both its square and any square in the same row and column that it can see, where only walls and the edges of the orchard block its sight. A sample layout (not the layout of the actual orchard, which is not given) is shown below.

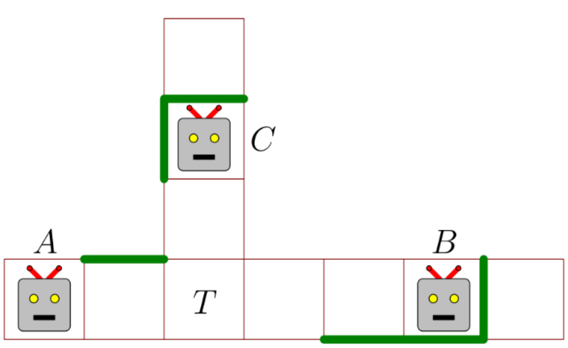


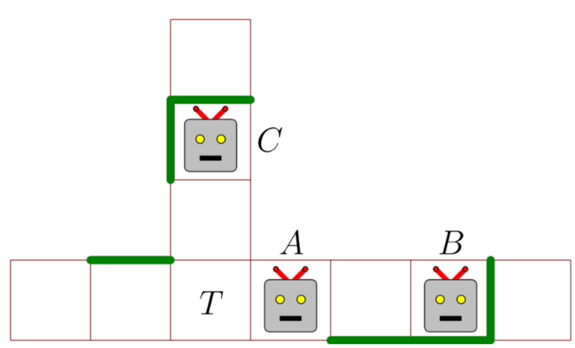
Although a square may be guarded by multiple sentries, the sentries have not been programmed to avoid attacking other sentries. Thus, no sentry may be placed on a square guarded by another sentry. The orchard's expert has found a way to guard all the squares of the orchard by placing 1000 sentries. However, the contractor shipped 2020 sentries. Show that it is impossible for the orchard to place all 2020 of the sentries without two of them attacking each other.

Solution

We show that if some square is guarded by 3 or more sentries, there must be two sentries that attack each other.

Take some square T that is guarded by 3 or more sentries. 2 of these sentries, A and B, must be in the same row or the same column, WLOG the same row. There is no wall between A and T or between B and T, so there is no wall between A and B either. Therefore A is guarding B's square and B is guarding A's square, and these two sentries will attack each other.





Suppose all 2020 sentries are placed in the orchard. We can place a human at each of the 1000 squares found by the orchard expert, so that every square of the orchard can be seen by a human. There must be at least one human who can see at least 3 robot sentries, or there would be at most $1000 \cdot 2 = 2000$ robot sentries in the orchard. But if the human can see the robot, the robot can also see the human. So there are at least 3 robot sentries guarding the human's square, and by the claim above, two of these sentries will attack each other. $\square$

3/3/31. A positive integer $n > 1$ is *juicy* if its divisors $d_1 < d_2 < \cdots < d_k$ satisfy $d_i - d_{i-1} | n$ for all $2 \le i \le k$. Find all squarefree juicy integers.

Solution 1: Brute Construction

First, we show that $d_i - d_{i-1}$ must divide d_{i-1} as well. If d_{i-1} and d_i are consecutive divisors of n, then so are $\frac{n}{d_i}$ and $\frac{n}{d_{i-1}}$. So $\frac{n}{d_{i-1}} - \frac{n}{d_i}$ must also be a divisor of n; in other words,

$$\frac{n}{\frac{n}{d_{i-1}} - \frac{n}{d_i}} = \frac{d_{i-1}d_i}{d_i - d_{i-1}}$$

is an integer. A prime p dividing $d_i - d_{i-1}$ must divide at least one of d_{i-1} and d_i, but since it divides their difference, p must divide both. Since $d_i - d_{i-1}$ is squarefree, it must then divide each of d_{i-1} and d_i.

$d_1 = 1$, which has only 1 as a divisor, so d_2 must be 2. Stopping here gives $n = 2$ as a valid solution.

d_3 must be 2 plus a divisor of 2 (since $d_3 - d_2$ must divide d_2), so d_3 it can be either 3 or 4. We are searching for squarefree n, so $d_3 = 3$. Then d_4 must be 6, and $n = 6$ is another valid solution.

$$1, 2, 3, 6$$

Continuing from here, we find $d_5 = 7, 8, 9, 12$. Of these, only 7 is squarefree. Then d_6 must be 14, since it cannot be 8. If $d_6 = 14$, $d_7 = 15, 16, 21, 28$. d_7 cannot be 15 because 5 is not a divisor of n, so the only remaining squarefree option is $d_7 = 21$. Then $d_8 = 22, 24, 28, 42$, of which only 42 is valid. Stopping here gives $n = 42$ as another solution.

$$1, 2, 3, 6, 7, 14, 21, 42$$

d_9 can be $43, 44, 45, 48, 49, 56$, or 63. Only 43 is squarefree, so $d_9 = 43$. Then d_{10} must be $43 + 43 = 86$, which means d_{11} can be $87, 88, 129$, or 172. Only 87 and 129 are squarefree, but 29 is not a divisor of n, so $d_{11} = 129$. Then d_{12} can be $130, 132, 172$, or 258. 258 is the only squarefree number in this list whose divisors all appear as an earlier divisor, so $d_{12} = 258$.

d_{12} has divisors $1, 2, 3, 6, 43, 86, 129, 258$, so d_{13} can be $259, 260, 261, 264, 301, 344, 387$, or 516. 259 is divisible by 37, 260 is divisible by 5, 261 by 9, 264, 344, and 516 by 4, and 387 by 9, so the only option is $d_{13} = 301$.

$d_{13} = 301$ has divisors $1, 7, 43, 301$, so d_{14} is $302, 308, 344$, or 602. 302 is divisible by 151 and 308 and 344 are both divisible by 4, so $d_{14} = 602$.

d_{14} has divisors $1, 2, 7, 14, 43, 86, 301$, and 602, so d_{15} can be $603, 604, 609, 616, 645, 688$, or 903. 903 is the only option.

d_{15} has divisors $1, 3, 7, 21, 43, 129, 301, 903$, so d_{16} must be $904, 906, 910, 924, 946, 1032, 1204$, or 1806. Only 1806 is squarefree and has all its divisors appearing earlier in the list, so $d_{16} = 1806$. Stopping here gives $n = 1806$ as another valid solution.

$$1, 2, 3, 6, 7, 14, 21, 42, 43, 86, 129, 258, 301, 602, 903, 1806$$

If we add any divisor of 1806 to 1806, we get either a composite number with a factor not dividing 1806, or a number that is not squarefree. So it is impossible to continue extending the divisor list, and the possibilities are $\boxed{n = 2, 6, 42, 1806.}$

Solution 2: General

We clean up the first solution and explain some of the patterns we found while constructing the sequence of divisors.

Again, we first show that $d_i - d_{i-1}$ must divide d_{i-1} as well. If d_{i-1} and d_i are consecutive divisors of n, then so are $\frac{n}{d_i}$ and $\frac{n}{d_{i-1}}$. So $\frac{n}{d_{i-1}} - \frac{n}{d_i}$ must also be a divisor of n; in other words,

$$\frac{n}{\frac{n}{d_{i-1}} - \frac{n}{d_i}} = \frac{d_{i-1}d_i}{d_i - d_{i-1}}$$

is an integer. A prime p dividing $d_i - d_{i-1}$ must divide at least one of d_{i-1} and d_i, but since it divides their difference, p must divide both. Since $d_i - d_{i-1}$ is squarefree, it must then divide each of d_{i-1} and d_i.

$d_1 = 1$ which has only 1 as a divisor, so d_2 must be 2. Now we build up the sequence of primes dividing n. We claim by induction that

$$p_k = \prod_{i=1}^{k-1} p_i + 1.$$

The empty product is 1 and we already have $p_1 = 2$, so the base case is satisfied.

Because $p_k = d_i = d_{i-1} + d$ where $d | d_{i-1}$, d_i can only be a prime when $d = 1$. We claim d_{i-1} must be $p_1 p_2 \cdots p_{k-1}$. Suppose for the sake of contradiction that p_r is the smallest prime not dividing d_{i-1}, for $r < k$. Then looking mod p_r, we have

$$p_j \equiv 1 \bmod p_r$$

for $j > r$, by construction. We also have

$$\prod_{j=1}^{r-1} p_j \equiv -1 \bmod p_r,$$

so

$$d_{i-1} + 1 \equiv 0 \bmod p_r,$$

contradicting that d_i is prime since $p_r | p_k$. So p_k must be $p_1 \cdots p_{k-1} + 1$.

Furthermore we claim if $d_{i-1} = p_1 \cdots p_{k-1}$, d_i must be p_k if it exists. Suppose for the sake of contradiction that p_s is the smallest prime dividing $d = d_i - d_{i-1}$. Then by the same logic as before, $\frac{p_1 \cdots p_{k-1}}{d} + 1$ is divisible by p_s. But $d_{i-1} + d = d\left(\frac{p_1 \cdots p_{k-1}}{d} + 1\right)$ is then divisible by p_s^2, contradiction. Thus the sequence of primes dividing n satisfies

$$p_k = \prod_{i=1}^{k-1} p_i + 1$$

as desired.

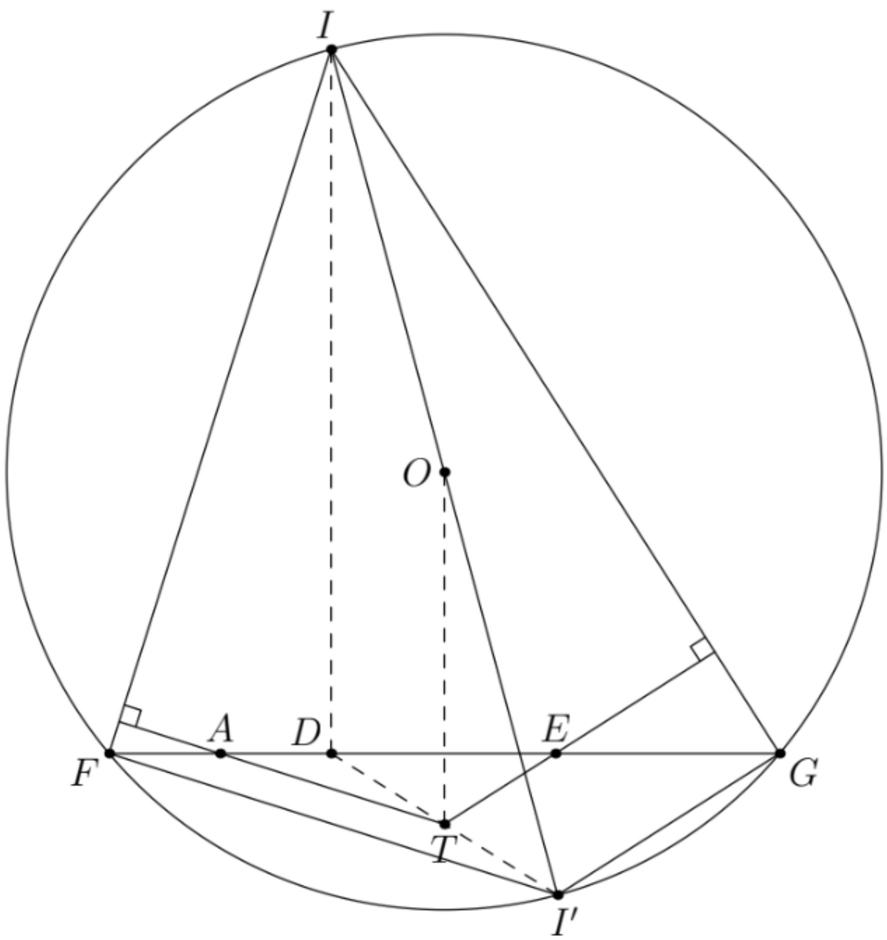
Now we can compute this sequence of primes. $p_1 = 2, p_2 = 3, p_3 = 7, p_4 = 43$, and $42 \cdot 43 + 1 = 1807$ is not prime (divisible by 13). So the possible values of n are given by the partial products of this sequence, or $\boxed{2, 6, 42, 1806.}$

4/3/31. Let FIG be a triangle and let D be a point on $\overline{FG}$. The line perpendicular to $\overline{FI}$ passing through the midpoint of $\overline{FD}$ and the line perpendicular to $\overline{IG}$ passing through the midpoint of $\overline{DG}$ intersect at T. Prove that $FT = GT$ if and only if $\overline{ID}$ is perpendicular to $\overline{FG}$.

Solution

Let I' be the point on the circumcircle of $\triangle FIG$ that is diametrically opposite point I, and let O be the circumcenter of $\triangle FIG$. Let A be the midpoint of $\overline{FD}$ and E be the midpoint of $\overline{DG}$. The homothety centered at D with scale factor 2 sends the line through A perpendicular to $\overline{FI}$ to the line through F perpendicular to $\overline{FI}$, and it sends the line through E perpendicular to $\overline{IG}$ to the line through G perpendicular to $\overline{GI}$. $\overline{I'F} \perp \overline{FI}$ and $\overline{I'G} \perp \overline{GI}$, so this homothety must send T to I'.

So $\overline{OT}$ is the midline of triangle $\overline{I'ID}$, and $\overline{OT} \parallel \overline{ID}$. $FT = GT$ iff T is also on the perpendicular bisector of $\overline{FG}$, or in other words, $\overline{OT} \perp \overline{FG}$. Since $\overline{OT} \parallel \overline{ID}$, this is true iff $\overline{ID} \perp \overline{FG}$ as desired. $\square$



5/3/31. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that for any reals x, y,

$$f(x+y)f(x-y) = (f(x))^2 - (f(y))^2.$$

Additionally, suppose that $f(x+2\pi) = f(x)$ and that there does not exist a positive real $a < 2\pi$ such that $f(x+a) = f(x)$ for all reals x. Show that for all reals x,

$$\left| f\left(\frac{\pi}{2}\right) \right| \geq f(x).$$

Solution

Let $P(x,y)$ denote the given assertion. By $P(0,0)$ we get

$$f(0)f(0) = (f(0))^2 - (f(0))^2 = 0,$$

so $f(0) = 0$. Comparing $P(x,y)$ and $P(x,-y)$, we get

$$f(x+y)f(x-y) = (f(x))^2 - (f(y))^2 = (f(x))^2 - (f(-y))^2,$$

so $f(y) = \pm f(-y)$ for each y. But by $P(0,y)$ we find

$$f(y)f(-y) = -(f(y))^2 \leq 0,$$

so $f(-y)$ must always be $-f(y)$. Thus f is odd.

Next we show $f(\pi) = 0$. From $P(x,\pi)$ and $f(x+2\pi) = f(x)$ we get

$$f(x+\pi)f(x-\pi) = (f(x))^2 - (f(\pi))^2$$

$$(f(\pi))^2 = (f(x))^2 - (f(x+\pi))^2$$

and from $P(x+\pi,\pi)$ we get

$$f(x+2\pi)f(x) = (f(x+\pi))^2 - (f(\pi))^2$$

$$(f(\pi))^2 = (f(x+\pi))^2 - (f(x))^2.$$

This means $(f(\pi))^2 = -(f(\pi))^2$, or $(f(\pi))^2 = 0 \Rightarrow f(\pi) = 0$ as desired.

Now suppose $f(c) = 0$ for some value of $c < \pi$. Then by $P(x+c,x)$,

$$0 = f(2x+c)f(c) = f(x+c)^2 - f(x)^2,$$

so $f(x+c)^2 = f(x)^2$. Similarly, evaluating $P(x,x-c)$ gives us $f(x-c)^2 = f(x)^2$. Furthermore from $P(x,c)$,

$$f(x+c)f(x-c) = f(x)^2,$$

so $f(x+c)$ and $f(x-c)$ must have the same sign and $2c$ is a period of the function, contradiction. Thus there are no zeros between 0 and π, and generally no zeros except at multiples of π.

Finally from $P\left(\frac{\pi}{2}, x\right)$,

$$f\left(\frac{\pi}{2}+x\right)f\left(\frac{\pi}{2}-x\right) = f\left(\frac{\pi}{2}\right)^2 - f(x)^2.$$

When $x = 0$, this value is positive. As long as $|x| \leq \frac{\pi}{2}$, by continuity this value will always be nonnegative since f has no zeros except for multiples of π. This shows $\left|f\left(\frac{\pi}{2}\right)\right| \geq f(x)$ for all $x \in [0, \pi]$. We can extend to all real values x by periodicity and because f is an odd function, so $\left|f\left(\frac{\pi}{2}\right)\right| \geq f(x)\ \forall x \in \mathbb{R}$ as desired. $\square$